# Maximizing the Throughput of a Rotating *Seru* with Nonpreemptive Discrete Stations

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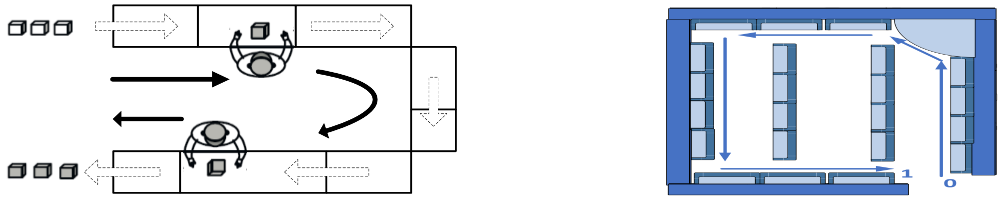
*Seru* production systems are an effective way to respond to ever-changing market demand. This paper focuses on maximizing the throughput of rotating *serus* with nonpreemptive stations, where a worker's operations cannot be disrupted. We analyze the effects of unbalanced worker velocities on non-value-added idle times. Through the use of dynamical system theories, we explicate the mechanism and dynamics of rotating *serus*, and identify the rules used to coordinate workers and distribute work content among stations to achieve the highest throughput. These findings provide practical guidelines for managers in floor shops for optimizing rotating *seru* design and maximizing throughput. Additionally, we explore the chaotic characteristics of rotating *serus* and simulate the effect of various factors on throughput. Finally, our comparative analysis demonstrates that the rotating *seru* offers a viable alternative to existing production systems to adapt to fluctuating demand.

***Keywords:*** Production systems; Bucket brigade; Work coordination; Order picking.

## 1. Introduction

A rotating *seru* is a type of production line that is staffed by two or more cross-trained workers. These lines are typically organized in a U-shaped layout. Each worker in a rotating *seru* is responsible for assembling a product from start to finish, carrying the product and moving it sequentially from station to station for value-added operations. After completing assembly at the final station, the worker then moves to the first station to begin assembling another product. As illustrated in Figure 1(a), the operation sequence of each worker is from left to right on the upper line, then from the top to the bottom on the vertical line, and finally from right to left on the bottom line. The minimum number of workers for a rotating *seru* is two; a single worker *seru* is known as a *yatai*. The dynamics of a rotating *seru* are similar to a game of rabbit-chasing, with each worker chasing the worker ahead of her/him and being chased by the worker behind her/him. When a faster worker catches up with a slower worker, the faster worker passes the slower worker. This paper examines methods for maximizing the throughput of a rotating *seru*.

*Seru* production systems are converted from traditional assembly lines to adapt to volatile markets characterized by frequently changing product models/types and fluctuating volume caused by short product life cycles and variable and surge demand. An assembly line can be dismantled and converted into a *seru* system that consists of one or more *serus*. According to Sull (2009), there are two approaches to doing business in a volatile market: companies can use agility to exploit emerging business opportunities or they can rely on tenaciousness to maintain the strength and stamina to weather market shifts. Sull recommended combining both approaches, which he called "agile tenaciousness", and suggested that an agile tenacious company can be created by breaking up a large organization into multiple independent, smaller profit-and-loss units. Transitioning to a *seru* system resembles Sull's framework: a large assembly line is broken up into multiple smaller, independent *serus*. The practice of *seru* production provides a way to create agile tenacious assembly systems.



(a). A rotating assembly *seru* (b). Component picking in a warehouse

Figure 1. Examples of production and order picking using rotating *serus*.

A key advantage of a rotating *seru* is its ability to flexibly handle significant fluctuations in production volume. For instance, if a *seru* system comprises several *yatais*, during a demand surge (such as occurs prior to shopping holidays), additional workers can be allocated to one or more *yatais* to increase production capacity. As a result, these *yatais* can be converted into rotating *serus* and then revert to *yatais* after the demand surge ends. See Kaizen.net (2021) for examples of such applications.

A rotating *seru* possesses several characteristics, as shown in Matrix 1. Some important concepts are defined as follows. A line is *discrete* if it is composed of distinct stations at which an item is assembled in a sequential manner. A station is *preemptive* if it can accommodate multiple workers at the same time; otherwise it is *nonpreemptive*. A worker has a *constant velocity* if she/he maintains the same pace (spends the same amount of time) at each station; otherwise the worker has *varied velocity*. The *work content*, or *labor content*, of a product type is defined as the total assembly time at all stations required to assemble a single item. The assembly time at a specific station is referred to as the work content of that station. We next give several examples.

Matrix 1. Characteristics of production and picking lines.

| | **Preemptive** | **Nonpreemptive** |
|---|---|---|
| **Constant velocity** | Constant, Preemptive | Constant, Nonpreemptive |
| **Varied velocity** | Varied, Preemptive | Varied, Nonpreemptive |

The use of rotating *serus* is prevalent in electronics factories, with companies such as Canon, Sony, Hitachi, Mitsubishi, Yaskawa, Pioneer, and Cosel reported to be using them in their operations (Matsuo, 2013). One example is a Japanese automobile component assembly factory in Kyushu, where the assembly process involves five distinct stations. This factory implemented rotating *serus* in 2011 to better handle frequent fluctuations in production volume. In the plants of Canon (Stecke et al., 2012;

Yin et al., 2008, 2017; Asao et al., 2014), rotating *serus* are often arranged in a U-shape or L-shape. Each worker performs all necessary tasks from start to finish without interruption. When a faster worker catches up with a slower worker, the faster worker passes the slower worker and continues working. Canon uses rotating *serus* not only in assembly operations, but also for order picking in their warehouses (see Figure 1(b)). Components are picked by two workers, with the layout and flow of components in the warehouse similar to that of the assembly *serus*. This approach to order picking is not unique to Canon; similar applications have been observed in the printing industry and in the plants of Washlet (electronic bidet) manufacturer TOTO (Hirasakura, 2019).

A practice used in the literature on production line analysis (Bartholdi and Eisenstein, 1996) is to standardize the work content of a product to one "time unit." For example, if the work content of a product is 36 minutes, it can be normalized to 1 unit by dividing 36 by 36. For constant velocity in Matrix 1, the work content of a product is evenly distributed throughout the production line. This is represented by normalizing the work content to one time unit, with the length of the production line also being equal to one. The length of each station corresponds to the percentage of work content performed there, and the position of an item on the production line represents the cumulative fraction of work content completed. For example, if a production line consists of three stations with lengths of 0.4, 0.3, and 0.3 respectively, an item in the middle of station 2 has completed 0.55 (0.4 + 0.15) of its work content. The length (i.e., work content) of each station can be readjusted so that the velocity of a worker is constant throughout the production line (i.e., so that the worker spends the same amount of time completing the operation at every station).

In an order picking line, a worker picks required items from a bin at each station, in order. The picking velocity of a picker at a station is determined by the experience of the picker and the density of the station (the quantity of items in a bin divided by the size of the bin). Therefore, for a picking line that consists of stations of the same density, a worker's velocity is constant throughout the line and determined by her/his picking experience. The density of a station, or bin, can be adjusted by adding or removing items. The size of a bin determines its preemptive property, with a larger bin being able to accommodate multiple pickers and a smaller bin being unable to do so. These characteristics are summarized in the first row (constant velocity) of Matrix 1.

Production lines can be complex, as operations often vary from station to station. This can lead to varied velocities across stations. However, in some cases, a standard velocity is the same at each station. For example, in a design-for-assembly modular product, where different modules are assembled together, all modules have the same interface (one that complies with industrial standards), resulting in the same assembly operation at each station (Sekine, 2018). Bartholdi and Eisenstein (1996) showed that a production line can be perfectly balanced by allocating the work content evenly throughout the line. In other words, the work content (or length) of the stations can be adjusted to achieve constant velocity. This can be easily verified using Little's law. Perfect balance means that every station has the same processing time (since there is no bottleneck). Velocity is the reciprocal of the processing time. In terms of preemptive property, when a station cannot accommodate more than one worker (e.g., when there is only one tool at the station), it is nonpreemptive; otherwise it is preemptive. These characteristics are summarized in the first row (constant velocity) of Matrix 1.

The second scenario in Matrix 1 (second row: varied velocity) is characterized by the inability to reallocate the work content of a station, resulting in varying velocities across different stations. This is commonly observed in production lines where operations are dissimilar across stations and workers possess varying skill levels for different tasks—for example, if worker 1 is faster than worker 2 for the operations in stations 1 and 2, but slower for the operation in station 3. Similarly, in order picking lines, varied velocity across stations is often attributed to varying bin density. These characteristics are summarized in the second row (varied velocity) of Matrix 1.

This paper is the first study to investigate the mechanism of a rotating *seru*. We focus on problems in the category of constant velocity and nonpreemptive. Problems that are nonpreemptive in nature are more challenging than those that are preemptive. Similarly, problems that involve varied velocity are more difficult to solve than those that involve constant velocity. Problems in some of the categories shown in Matrix 1 have been studied in the literature in the context of bucket brigades. For example, studies by Bartholdi and Eisenstein (1996) and Lim and Yang (2009) fall under the category of constant velocity and preemptive. McClain et al. (2000) is an example of a study that falls under the category of constant velocity and nonpreemptive. Wang et al. (2021) is an example of a study that falls under the category of varied velocity and preemptive. We were unable to find any publications that fall under the category of varied velocity and nonpreemptive, which is considered to be the most challenging category of research questions. Our study contributes to the literature in the category of constant velocity and nonpreemptive within the context of rotating *serus*. Other bucket brigade publications include those by Bartholdi and Eisenstein (2005), Armbruster and Gel (2006), Armbruster et al. (2007), Bartholdi et al. (2009), Lim (2011), Lim and Wu (2014), Lim (2017), Bukchin et al. (2018), and Cantor and Jin (2019).

## 2. Rotating *Serus* with Nonpreemptive Discrete Stations

In this section, we analyze rotating *serus* by establishing rotating *seru* rules and identifying the properties of rotating *serus*.

### 2.1. Rotating *serus* rules

We consider a rotating *seru* with $m$ stations and two workers, as depicted in Figure 2. We focus on the category of constant velocity and nonpreemptive, as shown in the upper right-hand quadrant of Matrix 1. The work content in this category is uniformly distributed across each station. To facilitate analysis, we normalize the work content of a rotating *seru* to 1, which also represents the length of the *seru*. The work content on each station (which is also the length of the station) is deterministic. Let $s_i$ $(i = 1,2,\dots,m)$ denote the work content on the $i$th station, so that $\sum_{i=1}^{m} s_i = 1$. Each worker continuously moves along a station as she/he progressively works on it. The initiation and completion of the rotating *seru* is represented by positions 0 and 1, respectively. The two workers' positions along the rotating *seru* can be viewed as the cumulative fraction of work content completed on their items. For example, if a worker is at position 0.65, this means that she/he has completed 65% of the assembly of the item in her/his hand. The two workers work forward with constant velocities of $v_1$ and $v_2$. We assume that worker 1 is faster than worker 2, that is $v_1 > v_2$.

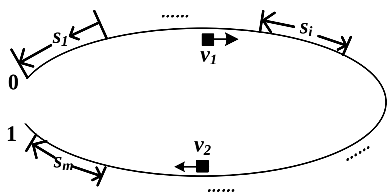



Figure 2. A rotating *seru* with $m$ stations and two workers.

Three rules that are used to coordinate workers are summarized as follows.

**<u>The Passing Rule:</u>**

When a faster worker is ready to work at a station, say station $i$, but it is occupied by a slower worker, the faster worker must wait until the slower worker completes her/his work at that station. Once the slower worker finishes, the faster worker passes the slower worker. They exchange items with each other. The faster worker then works at station $i + 1$. The slower worker works at station $i$ again.

**<u>The Blocking Rule:</u>**

If a slower worker is ready to work at a station that is occupied by a faster worker, the slower worker must wait until the faster worker completes her/his work at that station. Once the faster worker finishes, each worker will continue working on their own item.

**<u>The Circling Rule:</u>**

Once a worker completes an item at the exit of a *seru*, she/he relinquishes the completed item, walks across the aisle between the exit and the starting position of the *seru*, and initiates work on a new item.

These three rules, derived from observations of various rotating *seru* practices, provide a comprehensive understanding of most rotating *seru* behaviors. However, it is important to note that there are other rules (such as the use of buffers between stations, collaboration between workers on certain stations, and assistance from supervisors) that are not covered in this paper. These additional rules can be explored in future studies.

### 2.2. Behaviors and cycles

Our goal is to maximize the throughput of a rotating *seru* by reducing non-value-added idle times. To achieve this, we use tools from dynamical system theories, which are building blocks for chaos theory and self-organized processes. Many production line studies (e.g., Bartholdi and Eisenstein, 1996) have applied tools from dynamical systems to construct a self-organized line (i.e., a production line that can automatically balance itself). The concepts used in this study are explained as follows.

A 'state' refers to the vector of variables that describes a system at a particular point in time. For example, an atmospheric state can be represented by the vector $a$ consisting of three variables $a = (x, y, z)$, where $x$, $y$, and $z$ are temperature, pressure, and wind speed, respectively, at a specific time and location. The evolution of the state of the system over time is described by a system function $f$. For example, the state of the atmosphere at a specific time and location (e.g., March 3rd at 10 a.m. in Kyoto, Japan) can be represented by $a^{(0)} = \left(x^{(0)}, y^{(0)}, z^{(0)}\right)$ = (15c, 1013hPa, 20km/h). Using a state vector (e.g., $a^{(0)}$) as input, a system function (e.g., Lorenz atmospheric function $f$) can generate the subsequent state $a^{(1)} = \left(x^{(1)}, y^{(1)}, z^{(1)}\right)$ = (17c, 1015hPa, 15km/h) at a subsequent time point (e.g., March 4th at 10 a.m.) at the same location through $a^{(1)} = f\left(a^{(0)}\right)$. $a^{(0)}$ can be either deterministic (e.g., if $a^{(1)} = f\left(a^{(0)}\right)$ is calculated in the evening of March 3rd), or random (e.g., if $a^{(1)} = f\left(a^{(0)}\right)$ is calculated before March 3rd). It is important to note that although the system evolves in continuous time, the states are events at certain discrete time points, such as 10 a.m. every day for the atmospheric system. Additionally, the superscript $t = 0, 1, \ldots$ of the vector $a^{(t)}$ represents the index of states, not the time point. For example, $a^{(0)}$ and $a^{(1)}$ represent the initial and second states, recorded at 10 a.m. on March 3rd and 4th, respectively, not time points 0 and 1.

A function is a *map* if its domain space and range space are the same. Let $f$ be a map and $a^{(0)}$ be a state. The sequence of states

$$O(a^{(0)}) = \{a^{(0)}, a^{(1)} = f\left(a^{(0)}\right), \quad a^{(2)} = f\left(a^{(1)}\right) = f(f\left(a^{(0)}\right)), \\ a^{(3)} = f\left(a^{(2)}\right) = f(f(f\left(a^{(0)}\right))), \quad \ldots\}$$

is defined as the *orbit* of $a^{(0)}$ under $f$. We usually write $f^2\left(a^{(0)}\right), f^3\left(a^{(0)}\right), \ldots$ in place of $f\left(f\left(a^{(0)}\right)\right), f\left(f\left(f\left(a^{(0)}\right)\right)\right), \ldots$ . Therefore, we have

$$a^{(t+1)} = f\left(a^{(t)}\right) = f^{t+1}\left(a^{(0)}\right)$$

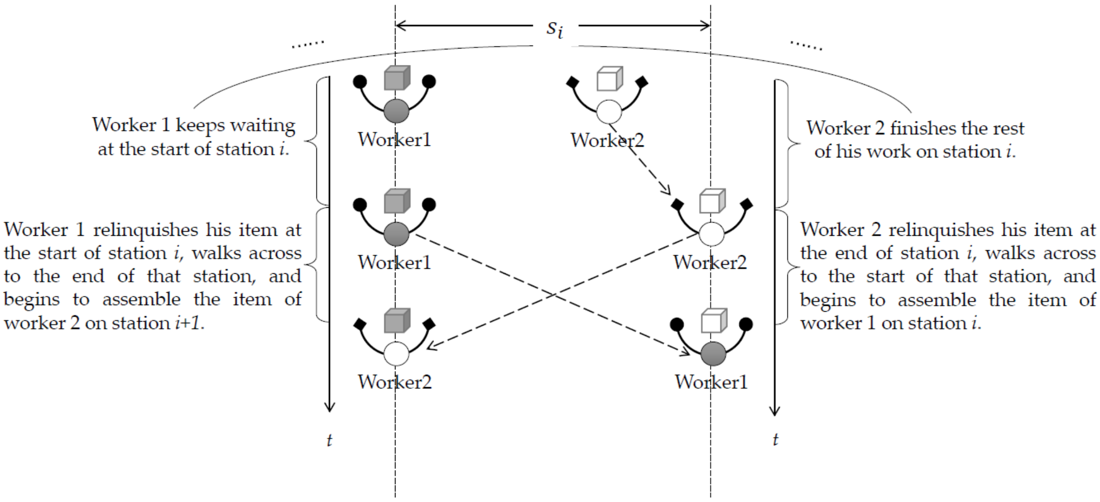
as before, where $t$ $(t = 0, 1, 2, \ldots)$ is the index of states that vary over time. The orbit of $a^{(0)}$ under $f$ can be written as $O(a^{(0)}) = \{a^{(0)}, f(a^{(0)}), f^2(a^{(0)}), f^3(a^{(0)}), \dots\}$.

In an orbit $O(a^{(0)})$, if there exists a state $a^{(t)}$ $(t \geq 0)$ and a positive integer $n$, such that $a^{(t)} = f^n(a^{(t)})$, then we say that the orbit $O(a^{(0)})$ has a period-$n$ cycle, and we call $a^{(t)}$ a periodic state of the period-$n$ cycle. For example, the orbit {.36, .78, .66, .45, .56, .45, .56, .45, .56, …} has a period-2 cycle, and .45 is the initial periodic state.

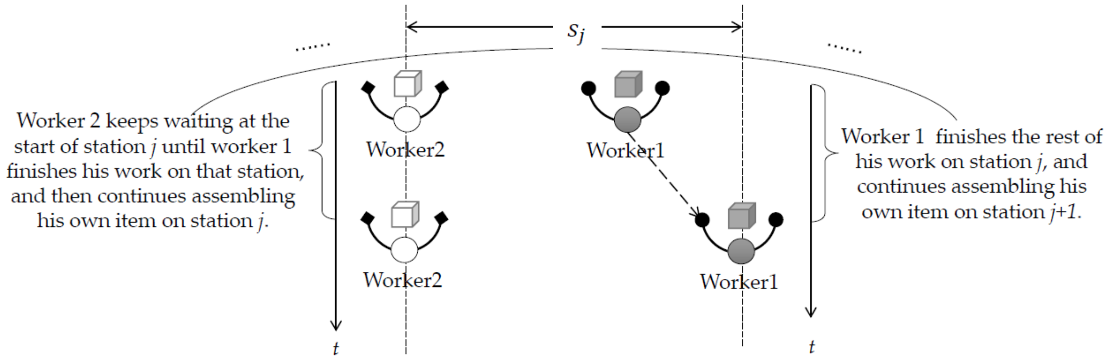
Next, we construct a rotating *seru* system using dynamical system theories as follows. Based on the rotating *seru* rules given in Section 2.1, we define three types of behaviors to model the dynamics of a rotating *seru* with nonpreemptive discrete stations.

- $P = \{p_1, p_2, \dots, p_m\}$: the set of passing behaviors. $p_i$ denotes that a pass occurs on station $i$ $(i = 1,2,\dots,m)$.
- $B = \{b_1, b_2, \dots, b_m\}$: the set of blocking behaviors. $b_j$ denotes that a block occurs on station $j$ $(j = 1,2,\dots,m)$.
- $C = \{c_1, c_2\}$: the set of circling behaviors. $c_k$ denotes that worker $k$ $(k = 1, 2)$ completes an item and initiates a new item.

It is important to note that the passing and circling behaviors apply to all stations within the assembly process. This means that these behaviors can occur at any station, including the final station. For example, when worker 2 is completing operations on station $m$, worker 1 may arrive at the beginning of that station. In this scenario, a passing event $p_m$ would occur, and based on the passing rule, worker 1 would have to wait for worker 2 to finish her/his work on station $m$ before they can exchange items. Once this exchange is completed, worker 1 would proceed to the end of station $m$, where a circling behavior $c_1$ would occur. Then, worker 2 would begin working on station $m$ again from the start.




(a) $p_i$, $i = 1,2 \dots, m$




(b) $b_j,\ j = 1,2\dots,m$

Figure 3. The movement of two workers incorporating passing and blocking.

Figure 3 illustrates the movement of two workers within a rotating *seru* that incorporates passing and blocking behaviors. Figure 3(a) and Figure 3(b) provide examples of passing and blocking on stations $i$ and $j$, respectively. The time it takes for a worker to walk between stations is not considered as the stations are located close to each other and the aisle is narrow.

Let $A = P \cup B$ denote the set of all passing and blocking behaviors. Let $f\colon A \to A$ be a map. $f$ determines the next state $a^{(t+1)} \in A$ based on the present state $a^{(t)} \in A$, that is $a^{(t+1)} = f\left(a^{(t)}\right)$. We study the dynamics of a rotating *seru* by analyzing the properties of its orbit $O(a^{(0)}) = \{a^{(0)}, a^{(1)}, \dots, a^{(t)}, \dots\} = \left\{a^{(t)} = f^{t}\left(a^{(0)}\right)\right\}_{t=0}^{\infty}$. This simplification frees us from having to worry about the details of the continuous time evolution of a rotating *seru*; instead, we can restrict our attention to all feasible states within an orbit. We find periodic cycles within an orbit as follows.

**THEOREM 1.** For any $m$-station two-worker rotating *seru* system, let $x_1$ and $x_2$ denote the arbitrary initial positions of workers 1 and 2, respectively; let $v_1$ and $v_2$ denote the work velocities of workers 1 and 2, respectively; and assume $v_1 > v_2$. There exist period-$n$ cycles $a^{(t)} = f^{n}\left(a^{(t)}\right)$ $(n \geq 1,\ t \geq 0)$ in the rotating *seru* system. For any combination of $(x_1, x_2, v_1, v_2)$, this cycle is unique.

All proofs are in Section A.1 of the online Appendix. Example 1 in Section A.2 of the Appendix illustrates Theorem 1.

Our main result is that, regardless of the initial positions (or initial states) of workers, the orbit of an $m$-station two-worker rotating *seru* becomes periodically locked in a sequence of passing and/or blocking behaviors. This is desirable because it creates positive effects such as improved efficiency due to the repetition of familiar work and better coordination among workers. Additionally, even if the workers' work velocities increase, the system can still maintain a periodic behavior without the need for management intervention.

The orbit $O(a^{(0)}) = \{a^{(0)}, a^{(1)}, \dots, a^{(t)}, \dots\}$ (where $a^{(t)} \in A$) under the system function $f\colon A \to A$ (where $A = P \cup B$) of a rotating *seru* records the evolution of passing and blocking behaviors. The focus of this paper is throughput, which is defined as the quantity of items completed during a time unit in a rotating *seru*. $O(a^{(0)})$ does not contain the information necessary for calculating throughput. A circling behavior $c_k$ denotes that worker $k$ $(k = 1, 2)$ completes an item and initiates work on a new item. Therefore, properties such as the frequency and sequence of circling behaviors $C = \{c_1, c_2\}$ between two successive states $a^{(t)}$ and $a^{(t+1)}$ $(t = 0, 1, ...)$ of an orbit $O(a^{(0)})$ can tell us the number of items completed by each worker.

For example, if $a^{(t)}a^{(t+1)}$ is present, it indicates that there is no item completion between states $a^{(t)}$ and $a^{(t+1)}$. On the other hand, if $a^{(t)}c_1c_2c_1a^{(t+1)}$ is present, it indicates that workers 1 and 2 completed 2 and 1 items respectively, between states $a^{(t)}$ and $a^{(t+1)}$, and that the sequence of item completion was: one by worker 1, then one by worker 2, and then another one by worker 1. By incorporating these circling behaviors into an orbit, we can calculate the throughput of a rotating *seru*. We can identify the frequency and sequence properties of these circling behaviors $C = \{c_1, c_2\}$ between two successive states $a^{(t)}$ and $a^{(t+1)}$ $(t = 0, 1, ...)$ of an orbit $O(a^{(0)})$ by using the following lemma.

**LEMMA 1.** Let $a^{(t)}$ $(t = 0, 1, ...)$ be an arbitrary state of an orbit $O(a^{(0)})$. If $c_1$ occurs between two successive states $a^{(t)}$ and $a^{(t+1)}$, we have the following:

(1) If $c_2$ also occurs between $a^{(t)}$ and $a^{(t+1)}$, the sequence of circling behaviors $C = \{c_1, c_2\}$ between $a^{(t)}$ and $a^{(t+1)}$ can only begin with $c_1$ (i.e., $a^{(t)}c_1 \ldots a^{(t+1)}$). In other words, the sequence pattern $a^{(t)}c_2 \ldots a^{(t+1)}$ is impossible.

(2) If $c_2$ also occurs between $a^{(t)}$ and $a^{(t+1)}$, $c_1$ and $c_2$ occur alternately between $a^{(t)}$ and $a^{(t+1)}$. This mean that two consecutive completions by the same worker is impossible. In other words, we always have $a^{(t)} \ldots c_1c_2 \ldots a^{(t+1)}$ or $a^{(t)} \ldots c_2c_1 \ldots a^{(t+1)}$ between $a^{(t)}$ and $a^{(t+1)}$; and $a^{(t)} \ldots c_1c_1 \ldots a^{(t+1)}$ and $a^{(t)} \ldots c_2c_2 \ldots a^{(t+1)}$ are impossible.

(3) If $a^{(t+1)} \in P$, the number of occurrences of $c_1$ is one more than the number of occurrences of $c_2$ between $a^{(t)}$ and $a^{(t+1)}$.

(4) If $a^{(t+1)} \in B$, the number of occurrences of $c_1$ is equal to the number of occurrences of $c_2$ between $a^{(t)}$ and $a^{(t+1)}$.

Let the number of occurrences of $c_2$ between $a^{(t)}$ and $a^{(t+1)}$ be $k$ $(k \geq 0)$. By Lemma 1, we can determine the circling behaviors between two successive states $a^{(t)}$ and $a^{(t+1)}$ as outlined in the following lemma.

**LEMMA 2.** Let $k$ be a nonnegative integer and $r$ be a velocity ratio $r = v_1/v_2$, where $r > 1$. The behaviors between two successive states $a^{(t)}$ and $a^{(t+1)}$ are:

(1) If $a^{(t+1)} \in P$, we have $a^{(t)} \underbrace{c_1c_2 \ldots c_1c_2}_{k} c_1 a^{(t+1)}$, where $k \leq \left\lceil \frac{m-1-r}{m(r-1)} \right\rceil + 1$;

(2) If $a^{(t+1)} \in B$, we have $a^{(t)} \underbrace{c_1c_2 \ldots c_1c_2}_{k} a^{(t+1)}$, where $k \leq 1$,

where $\underbrace{c_1c_2 \ldots c_1c_2}_{k}$ denotes that the circling behavior pair $c_1c_2$ iterate $k$ times.

In (1) of Lemma 2, $\lceil \alpha \rceil$ represents the rounded-up integer of α. For example, if α=0.2, we get $\lceil \alpha \rceil$=1. $k$ is the number of items completed by the second worker between two successive states $a^{(t)}$ and $a^{(t+1)}$. $k$ can range from 0 to $\left\lceil \frac{m-1-r}{m(r-1)} \right\rceil + 1$. To understand the relationship between $k$, $m$, and $r$, let $\alpha = \frac{m-1-r}{m(r-1)}$. We have $\frac{\partial \alpha}{\partial m} = \frac{r+1}{(r-1)m^2}$. Recall that $m \geq 3$ and $r = v_1/v_2 \geq 1$, so $\frac{\partial \alpha}{\partial m} > 0$ and α increases in $m$. We analyze the case of $m \gg r$ as follow. We have $\alpha \rightarrow \frac{1}{(r-1)}$. When $r \geq 2$, $\lceil \alpha \rceil = 1$ and $k = 0\ or\ 1\ or\ 2$. When $r < 2$, $\lceil \alpha \rceil = k^* > 1$, where $k^*$ is an integer and $k = 0 \text{ or } 1 \text{ or}, \ldots, k^* + 1$. Theoretically, if $r = 1$, $k$ can be $+\infty$, which means that $a^{(t+1)} \in P$ will never occur. This is the case of two workers having the same velocity, meaning that neither worker will ever pass (or be passed by) the other worker. The absence of passing eliminates the waste of waiting (see Rule for Passing) and results in high throughputs. This is the reason why managers, in practice, prefer to use workers with similar velocities to construct a rotating *seru*. Example 2 in Section A.2 of the online Appendix illustrates Lemmas 1 and 2.

Examples 1 and 2 illustrate that when worker velocities are fixed, which is typically the case in practice and not easy to change in a short period, adjusting the work content on stations can result in higher throughput. This observation aligns with the cases and literature (Bartholdi and Eisenstein, 1996; Sekine, 2018) discussed in Section 1. We summarize this observation in the following remark:

**Remark**: For constant velocity and nonpreemptive rotating *serus*, adjusting the work content on stations can yield the highest throughput when worker velocities are fixed.

In the following sections, we will apply this concept to find the highest throughput. Theorem 1 and Lemmas 1 and 2 in this section will be used to construct and analyze rotating *serus* with discrete stations in Sections 3 and 4.

## 3. Three-Station Two-Worker Rotating *Serus* with Nonpreemptive Discrete Stations

In this section, the special case of rotating *serus* with three stations and two workers is discussed. The distribution of work content on the three stations are $s_1$, $s_2$, and $s_3$, where $s_1 + s_2 + s_3 = 1$. The velocities of the two workers are $v_1$ and $v_2$, where $v_1 > v_2$. We analyze all scenarios of behaviors occurring in period-$n$ cycles and calculate their throughputs. The result will help us in the analysis of the $m$-station case. We define a velocity ratio $r = v_1/v_2$. There are two cases: (1) $r \geq 2$ and (2) $2 > r > 1$. The details of the two cases are analyzed as follows.

### 3.1. $r \geq 2$

Let $a^{(t)}$ $(t = 0, 1, \ldots)$ be an arbitrary element of an orbit $O(a^{(0)})$, while the function $f$ determines its next state $a^{(t+1)} = f\left(a^{(t)}\right)$. We will analyze $f$ for $r \geq 2$ with respect to different work content distributions on the three stations and different velocity ratios in the following Lemma 3. In short, $a^{(t)}$ and $a^{(t+1)}$ are independent and dependent variables of function $f$, respectively. $s_1$, $s_2$, $s_3$, and $r$ are parameters of function $f$.

**LEMMA 3.** The function $a^{(t+1)} = f(a^{(t)})$ for $r \geq 2$ is given as follows.

If $a^{(t)} \in \{p_1, b_1\}$,

$$a^{(t+1)} = \begin{cases} p_1, & if\ s_1 \geq 1/(r+1); \\ b_2, & if\ rs_1 < s_2; \\ b_3, & if\ rs_1 \geq s_2, rs_1 + (r-1)s_2 < s_3,; \\ p_2, & if\ rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3, s_1 < 1/(r+1), s_3 \leq (r-1)/r; \\ p_3, & otherwise. \end{cases}$$

If $a^{(t)} \in \{p_2, b_2\}$,

$$a^{(t+1)} = \begin{cases} p_2, & if\ s_2 \geq 1/(r+1); \\ b_3, & if\ rs_2 < s_3; \\ b_1, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 < s_1; \\ p_3, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_2 < 1/(r+1), s_1 \leq (r-1)/r; \\ p_1, & otherwise. \end{cases}$$

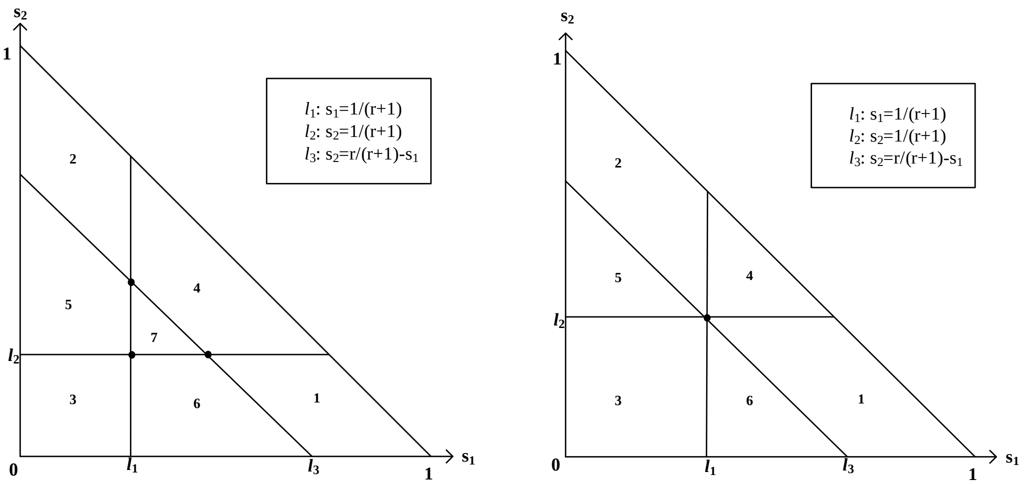
If $a^{(t)} \in \{p_3, b_3\}$,

$$a^{(t+1)} = \begin{cases} p_3, & if\ s_3 \geq 1/(r+1); \\ b_1, & if\ rs_3 < s_1; \\ b_2, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 < s_2; \\ p_1, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_3 < 1/(r+1), s_2 \leq (r-1)/r; \\ p_2, & otherwise. \end{cases}$$

By Lemma 3, we have $a^{(t+1)} = f(a^{(t)})$ for any input of $a^{(t)}$. By Lemma 2, we have a circling behavior path of function $f$ between $a^{(t)}$ and $a^{(t+1)}$. For example, if $a^{(t+1)} \in P$, we have $a^{(t)} \underbrace{c_1 c_2 \ldots c_1 c_2}_{k} c_1 a^{(t+1)}$. According to Theorem 1, there is a unique period-*n* cycle $a^{(t)} = f^n\left(a^{(t)}\right)$. (See Figure A.4 (a) and (b) of Example 1 for a period-1 cycle.) Unfortunately, Lemmas 2 and 3 do not tell us how to find the unique period-*n* cycle $a^{(t)} = f^n\left(a^{(t)}\right)$ (as suggested by Theorem 1) for a three-station two-worker rotating *seru* with a specific work content distribution $(s_1, s_2, s_3)$ and given initial worker positions $(x_1, x_2)$. We next discuss how to find the unique period-*n* cycle $a^{(t)} = f^n\left(a^{(t)}\right)$, and its behavior paths and throughputs. The results are in Lemma 4.

Recall that the function $a^{(t+1)} = f(a^{(t)})$ has two variables $a^{(t)}$ and $a^{(t+1)}$, and four parameters $s_1$, $s_2$, $s_3$, and $r$. Lemma 3 and Example 1 show that different parameter values generate different $a^{(t+1)}$ for the same input of $a^{(t)}$. To illustrate all possible values of the four parameters, we use Figure 4.

In Figure 4, the horizontal and vertical axes, both ranging from 0 to 1, correspond to $s_1$ and $s_2$, respectively. The triangle region describes all feasible work content distributions $(s_1, s_2, s_3)$. That is, a point $(s_1, s_2)$ in the region represents a distribution of work content on the stations, $s_1$, $s_2$, and $s_3 = 1 - (s_1 + s_2)$. For example, the point (0.3, 0.4) means a work content distribution of (0.3, 0.4, 0.3). The work content region of the triangle is $s_2 < 1 - s_1$. The three vertexes correspond to (0, 0, 1), (1, 0, 0), and (0, 1, 0), respectively.




(a) $r > 2$ (b) $r = 2$

Figure 4. The work content regions for $r \geq 2$.

Besides $(s_1, s_2, s_3)$, the value of the velocity ratio $r = v_1/v_2$ is the fourth necessary parameter to be used to finalize the behavior path of $a^{(t)} = f^n(a^{(t)})$ for given arbitrary initial positions of two workers $x_1$ and $x_2$. The detailed process is as follows. As in Example 1 in Figure A.4, with specific $(s_1, s_2, s_3)$, $x_1$, $x_2$, and $r = v_1/v_2$, we can identify the initial state $a^{(0)}$. With Lemma 3, we find $a^{(1)} = f(a^{(0)})$. With Lemmas 1 and 2, we obtain the behavior path $a^{(0)}\underbrace{c_1c_2 \dots c_1c_2}_{k}c_1a^{(1)}$ or $a^{(0)}\underbrace{c_1c_2 \dots c_1c_2}_{k}a^{(1)}$ between $a^{(0)}$ and $a^{(1)}$. This process continues for $a^{(2)}$, $a^{(3)}$, and so on. According to Theorem 1, we will finally find a period-$n$ cycle $a^{(t)} = f^n(a^{(t)})$ that holds a specific behavior path (see details in the online proof of Theorem 1). The result is summarized in the following Lemma 4.

In Lemma 4, for three-station two-worker rotating *serus* with $r \geq 2$, there are three different period-1 cycles corresponding to seven (six) different parameter values, which can be depicted as seven (six) regions in Figure 4. Three lines $s_1 = 1/(r+1)$, $s_2 = 1/(r+1)$, and $s_1 + s_2 = r/(r+1)$, partition the triangle into seven and six mutually exclusive regions for $r > 2$ and $r = 2$, respectively. Each region is a specific combination of four parameters $s_1$, $s_2$, $s_3$, and $r$. For example, the domains of the four parameters in region 7 of Figure 4(a) are $s_1 > 1/(r+1)$, $s_2 > 1/(r+1)$, $s_3 < 1 -$

$2/(r+1)$, and $(r+1) > max\{1/s_1, 1/s_2, 2/(s_1+s_2)\}$. For Figure 4(b), $r$ decreases to 2, so $l_1$ and $l_2$ move right and up, respectively. Three lines intersect at the same point, resulting in the disappearance of region 7. For convenience, we define the three cycles with their behavior paths as $\theta_1 = p_1c_1p_1$, $\theta_2 = p_2c_1p_2$, and $\theta_3 = p_3c_1p_3$; and their corresponding three throughputs as $\tau_1 = v_2/s_1$, $\tau_2 = v_2/s_2$, and $\tau_3 = v_2/s_3$.

**LEMMA 4.** If $r \geq 2$, a two-worker three-station rotating *seru* has period-1 cycles and its throughputs are as follows.

Region $i$: The cycle is $\theta_i$, and its throughput is $\tau_i$, $i = 1, \ldots, 3$.

Region 4: If $a^{(0)} \in \{p_1, p_3, b_1, b_3\}$, the cycle is $\theta_1$, and the throughput is $\tau_1$. If $a^{(0)} \in \{p_2, b_2\}$, the cycle is $\theta_2$, and the throughput is $\tau_2$.

Region 5: If $a^{(0)} \in \{p_1, p_2, b_1, b_2\}$, the cycle is $\theta_2$, and the throughput is $\tau_2$. If $a^{(0)} \in \{p_3, b_3\}$, the cycle is $\theta_3$, and the throughput is $\tau_3$.

Region 6: If $a^{(0)} \in \{p_2, p_3, b_2, b_3\}$, the cycle is $\theta_3$, and the throughput is $\tau_3$. If $a^{(0)} \in \{p_1, b_1\}$, the cycle is $\theta_1$, and the throughput is $\tau_1$.

Region 7: If $a^{(0)} \in \{p_1, b_1\}$, the cycle is $\theta_1$, and the throughput is $\tau_1$. If $a^{(0)} \in \{p_2, b_2\}$, the cycle is $\theta_2$, and the throughput is $\tau_2$. If $a^{(0)} \in \{p_3, b_3\}$, the cycle is $\theta_3$, and the throughput is $\tau_3$. If $r = 2$, region 7 does not exist.

In the above lemma, region 1 corresponds to more work content on station 1 than on stations 2 and 3. Worker 2 cannot finish her/his work on station 1 before worker 1 completes an item and returns to the start of station 1. The cycle behavior path is $\theta_1 = p_1c_1p_1$ and the generated throughput is $\tau_1 = (s_1/v_2)^{-1} = v_2/s_1$. (See details in the online Appendix proofs.) Similarly, regions 2 and 3 correspond to more work content on stations 2 and 3. Worker 2 cannot finish her/his work on station 2 or 3 before worker 1 completes an item and returns to the start of station 2 or 3. Regions 4, 5, and 6 correspond to relatively less work content on stations 3, 1, and 2, respectively. Each of these three regions has two feasible cycles that depend on the initial state $a^{(0)}$, which is the result of different initial worker positions $x_1$ and $x_2$ (see Example 1). Region 7 corresponds to relatively equal work content on station 1, station 2, and station 3. There are three feasible cycles that depend on the initial state. Example 3 in Section A.2 of the online Appendix illustrates Lemmas 3 and 4.

The throughput of 2.0 in Example 3 is not the maximal throughput this *seru* can achieve. Recall the observation in Section 2 that the work content on the stations can be adjusted to achieve maximal throughput. With Lemmas 3 and 4, we are able to express this idea in Theorem 2, which provides the conditions for achieving the maximal throughput for various initial states.

**THEOREM 2**. If $r \geq 2$, a two-worker three-station rotating *seru* achieves its maximal throughput $v_1 + v_2$ as follows.

If $a^{(0)} \in \{p_1, b_1\}$, the maximal throughput is achieved by either of

(1) $s_1 = \frac{1}{r+1}$; or (2) $s_2 = \frac{1}{r+1}$ and $0 < s_1 < \frac{1}{r+1}$.

If $a^{(0)} \in \{p_2, b_2\}$, the maximal throughput is achieved by either of

(1) $s_2 = \frac{1}{r+1}$; or (2) $s_3 = \frac{1}{r+1}$ and $0 < s_2 < \frac{1}{r+1}$.

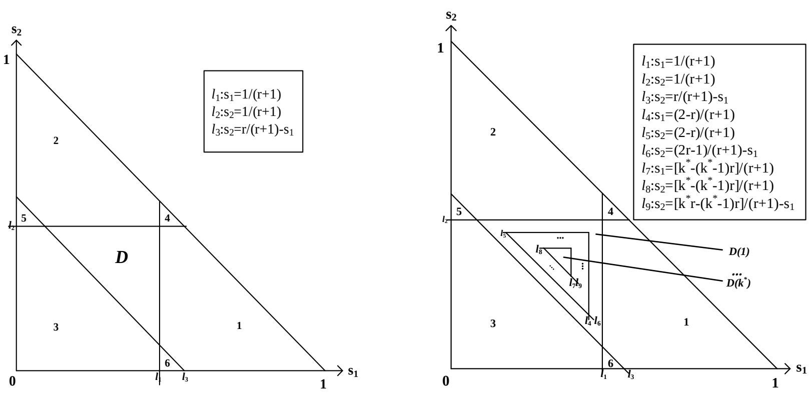
If $a^{(0)} \in \{p_3, b_3\}$, the maximal throughput is achieved by either of

(1) $s_3 = \frac{1}{r+1}$; or (2) $s_1 = \frac{1}{r+1}$ and $0 < s_3 < \frac{1}{r+1}$.

In Example 4 (Section A.2 of the online Appendix), we demonstrate all of the possible throughputs that can be achieved within a given work content space, including the maximal throughput described in Theorem 2. Specifically, we look at the values of $(s_1, s_2, s_3 = 1 - s_1 - s_2)$ and their impact on the overall performance of the rotating *seru*.

### 3.2. $2 > r > 1$

In the case where $2 > r > 1$, the cycles and throughputs in regions 1-6 of Figure 4(a) are the same as the case where $r \geq 2$. As the velocity ratio $r$ decreases, worker 2 does not continue working on a single station. Region 7 becomes a more complex area, known as region $D$, where $s_1 < 1/(r+1)$, $s_2 < 1/(r+1)$, and $s_3 < 1/(r+1)$, as shown in Figure 5(a). In region $D$, there is only a small difference in the work content at different stations; e.g., $(s_1, s_2, s_3) = (1/3, 1/3, 1/3)$. The workers have opportunities to complete more items between two successive states $a^{(t)}$ and $a^{(t+1)}$. By (1) of Lemma 2, between two successive states $a^{(t)}$ and $a^{(t+1)}$, the number of items completed by the second worker is $k$, which can be any number ranging from 0 to $\left\lceil \frac{m-1-r}{m(r-1)} \right\rceil + 1$. For a three-station *seru* ($m = 3$), let $k^* = \left\lceil \frac{m-1-r}{m(r-1)} \right\rceil = \left\lceil \frac{\mathbf{2-r}}{\mathbf{3(r-1)}} \right\rceil$. The dynamics of region $D$ can be characterized by $k^*$.




(a) Region $D$ arises. (b) $k^*$ layers.

Figure 5. The work-content region $D$ is partitioned into $k^*$ layers when $2 > r > 1$.

Region $D$ is partitioned into $k^*$ layers, as shown in Figure 5(b). $T(k)$ is a sub region defined as $s_i < \frac{k-(k-1)r}{r+1}$ for $i = 1, 2$. When $k = 1, 2, \dots, k^* - 1$, the $k$-th layer is denoted as $T(k+1) \le D(k) < T(k)$. The $k^*$-th layer is the innermost triangle and is denoted as $D(k^*) = T(k^*)$. As region $D$ is the only difference from the case of $r \ge 2$, we study the dynamics of region $D$ through the $k^*$ layers. There are two cases: $k = 1,2 \dots, k^* - 1$ and $k = k^*$.

- **<u>Case 1: $k = 1, 2 \dots, k^* - 1$</u>**

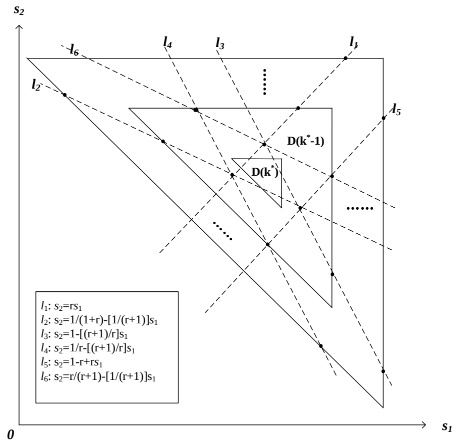



Figure 6. Layer $D(k^* - 1)$.

As shown in Figure 6, for $k \ge 1$ or $k^* \ge 2$, each Layer $D(k)$, $k = 1,2, \dots, k^* - 1$, is partitioned by lines $l_1 \sim l_6$. Recall that each sub region represents a set of work content $(s_1, s_2, s_3)$. We analyze the cycles and throughputs for the case $k = k^* - 1$ because layer $D(k^* - 1)$ is partitioned into more sub regions than any other layer; i.e., $D(k^* - 1)$ is the most complex and dynamic layer. Analyses for cases of $k = 1,2, \dots, k^* - 2$ are similar.

The system function $a^{(t+1)} = f\left(a^{(t)}\right)$ for $D(k^* - 1)$ with respect to different work content distributions is given by the following lemma. Let

$\Phi(k) = \frac{k-(k-1)r}{r+1}$ and $\Omega(k) = \frac{k(r-1)}{r}$, where $k = 1,2, \dots, k^* - 1$.

**<u>LEMMA 5.</u>** The function $a^{(t+1)} = f(a^{(t)})$ for $D(k)$, $k = k^* - 1$, is given as follows.

If $a^{(t)} \in \{p_1, b_1\}$

$$a^{(t+1)} = \begin{cases} b_2, & if\ rs_1 < s_2; \\ b_3, & if\ rs_1 \ge s_2, rs_1 + (r-1)s_2 < s_3; \\ p_2, & if\ rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 \le \Omega(k)\ or \\ & \quad rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 > \Omega(k), s_2 > r\Omega(k), s_1 < \Phi(k+1), s_2 \ge \Phi(k+1); \\ p_3, & if\ rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 > \Omega(k), s_2 \le r\Omega(k)\ or \\ & \quad rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 > \Omega(k), s_2 > r\Omega(k), s_1 < \Phi(k+1), s_2 < \Phi(k+1), s_3 \ge \Phi(k+1); \\ p_1, & if\ rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 > \Omega(k), s_2 > r\Omega(k), s_1 \ge \Phi(k+1). \end{cases}$$

If $a^{(t)} \in \{p_2, b_2\}$

$a^{(t+1)}$

$$= \begin{cases} b_3, & if\ rs_2 < s_3; \\ b_1, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 < s_1; \\ p_3, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_1 \leq \Omega(k)\ or \\ & \quad rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_1 > \Omega(k), s_3 > r\Omega(k), s_2 < \Phi(k+1), s_3 \geq \Phi(k+1); \\ p_1, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_1 > \Omega(k), s_3 \leq r\Omega(k)\ or \\ & \quad rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_1 > \Omega(k), s_3 > r\Omega(k), s_2 < \Phi(k+1), s_3 < \Phi(k+1), s_1 \geq \Phi(k+1); \\ p_2, & if\ rs_2 \geq s_3, rs_2 + (r-1)s_3 \geq s_1, s_1 > \Omega(k), s_3 > r\Omega(k), s_2 \geq \Phi(k+1). \end{cases}$$

If $a^{(t)} \in \{p_3, b_3\}$

$a^{(t+1)}$

$$= \begin{cases} b_1, & if\ rs_3 < s_1; \\ b_2, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 < s_2; \\ p_1, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_2 \leq \Omega(k)\ or \\ & \quad rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_2 > \Omega(k), s_1 > r\Omega(k), s_3 < \Phi(k+1), s_1 \geq \Phi(k+1); \\ p_2, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_2 > \Omega(k), s_1 \leq r\Omega(k)\ or \\ & \quad rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_2 > \Omega(k), s_1 > r\Omega(k), s_3 < \Phi(k+1), s_1 < \Phi(k+1), s_2 \geq \Phi(k+1); \\ p_3, & if\ rs_3 \geq s_1, rs_3 + (r-1)s_1 \geq s_2, s_2 > \Omega(k), s_1 > r\Omega(k), s_3 \geq \Phi(k+1). \end{cases}$$

Lemma 5 is similar to Lemma 3 for the $r \geq 2$ case. Based on Theorem 1 and using results of the above lemma, for the combination of a given work content distribution (i.e., $s_1$, $s_2$, and $s_3$) and a given initial state $a^{(0)}$, a unique period-$n$ cycle $a^{(t)} = f^n(a^{(t)})$ can be obtained. As in the case of $r \geq 2$, the behavior path between two consecutive states $a^{(t)}$ and $a^{(t)}$, and the throughput of a period-$n$ cycle $a^{(t)} = f^n(a^{(t)})$ can be found in the following Lemma 6, which is similar to Lemma 4 of the $r \geq 2$ case. All analyses are similar to Lemmas 3 and 4. We keep our descriptions here as simple as possible. For details, see the proofs of Lemmas 5 and 6 in the Appendix and the analyses of Lemmas 3 and 4.

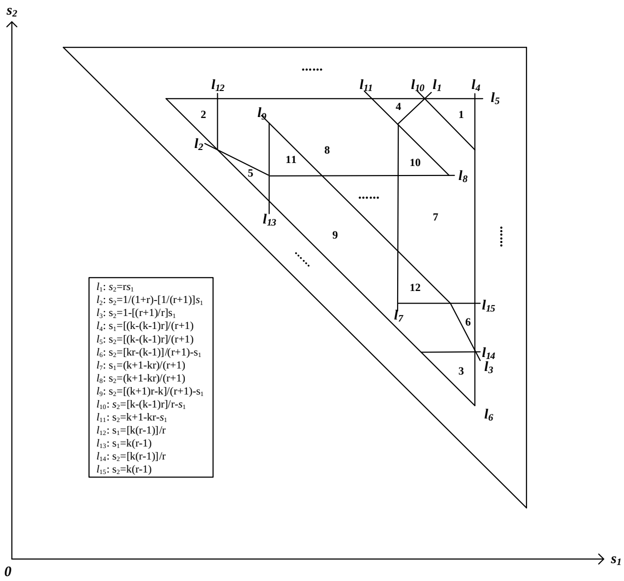



Figure 7. $D(k^* - 1)$ is partitioned into 12 sub regions.

There are twelve work content distributions (sub regions) corresponding to nine cycles (behavior paths) and throughputs (see Figure 7 for details). For convenience, we define the nine cycles and throughputs in Table 1.

Table 1. Cycles (Behavior Paths) and Throughputs of $k^*-1$

| Cycles (Behavior Paths) | Throughputs |
|---|---|
| $\theta_1(k) = p_1 \underbrace{c_1c_2 \dots c_1c_2}_{k-1} c_1p_2 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_1$ | $\tau_1(k) = \frac{4k}{2k-s_3} \times v_2$ |
| $\theta_2(k) = p_2 \underbrace{c_1c_2 \dots c_1c_2}_{k-1} c_1p_3 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_2$ | $\tau_2(k) = \frac{4k}{2k-s_1} \times v_2$ |
| $\theta_3(k) = p_1 \underbrace{c_1c_2 \dots c_1c_2}_{k-1} c_1p_3 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_1$ | $\tau_3(k) = \frac{4k}{2k-s_2} \times v_2$ |
| $\theta_4(k) = p_1b_2 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_1$ | $\tau_4(k) = \frac{2k+1}{k+s_2/r} \times v_2$ |
| $\theta_5(k) = p_2b_3 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_2$ | $\tau_5(k) = \frac{2k+1}{k+s_3/r} \times v_2$ |
| $\theta_6(k) = p_3c_1c_2b_1 \underbrace{c_1c_2 \dots c_1c_2}_{k-1} c_1p_3$ | $\tau_6(k) = \frac{2k+1}{k+s_1/r} \times v_2$ |
| $\theta_7(k) = p_1 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_1$ | $\tau_7(k) = \frac{2k+1}{k+s_1} \times v_2$ |
| $\theta_8(k) = p_2 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_2$ | $\tau_8(k) = \frac{2k+1}{k+s_2} \times v_2$ |
| $\theta_9(k) = p_3 \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_3$ | $\tau_9(k) = \frac{2k+1}{k+s_3} \times v_2$ |

**<u>LEMMA 6.</u>** If $2 > r > 1$, for $D(k^*-1)$, a two-worker three-station rotating *seru* has distinct period-*n* cycles and throughputs in each of the following twelve regions.

Region $k_i$: The cycle is $\theta_i(k)$, and its throughput is $\tau_i(k)$, $i = 1, \dots, 9$.

Region $k_{10}$: If $a^{(0)} \in \{p_1, p_3, b_1, b_3\}$, the cycle is $\theta_7(k)$ with throughput of $\tau_7(k)$; if $a^{(0)} \in \{p_2, b_2\}$, the cycle is $\theta_8(k)$ with throughput of $\tau_8(k)$.

Region $k_{11}$: If $a^{(0)} \in \{p_1, p_2, b_1, b_2\}$, the cycle is $\theta_8(k)$ with throughput of $\tau_8(k)$; if $a^{(0)} \in \{p_3, b_3\}$, the cycle is $\theta_9(k)$ with throughput of $\tau_9(k)$.

Region $k_{12}$: If $a^{(0)} \in \{p_2, p_3, b_2, b_3\}$, the cycle is $\theta_9(k)$ with throughput of $\tau_9(k)$; if $a^{(0)} \in \{p_1, b_1\}$, the cycle is $\theta_7(k)$ with throughput of $\tau_7(k)$.

- **<u>Case 2: $k = k^*$</u>**

For layer $D(k^*)$, there are a maximum of 15 sub regions based on the different values of velocity ratios. Besides cycles $\theta_1(k)$- $\theta_9(k)$, there are two more cycles and throughputs as follows.

$$\theta_0(k^*) = p_1 \underbrace{c_1c_2 \dots c_1c_2}_{k^*-1} c_1p_2 \underbrace{c_1c_2 \dots c_1c_2}_{k^*-1} c_1p_3 \underbrace{c_1c_2 \dots c_1c_2}_{k^*} c_1p_1, \ \tau_0(k^*) = \frac{6k^*-1}{3k^*-1} \times v_2;$$

$$\theta_0'(k^*) = p_1 \underbrace{c_1c_2 \dots c_1c_2}_{k^*-1} c_1p_3 \underbrace{c_1c_2 \dots c_1c_2}_{k^*} c_1p_2 \underbrace{c_1c_2 \dots c_1c_2}_{k^*} c_1p_1, \ \tau_0'(k^*) = \frac{6k^*+1}{3k^*} \times v_2.$$

In the following lemma, we show cycles and throughputs of each sub region. For multiple figures that illustrate details, see the e-companion online file of the Appendix.

**<u>LEMMA 7.</u>** If $2 > r > 1$, for $D(k^*)$, a two-worker three-station rotating *seru* has distinct period-$n$ cycles and throughputs in each of the following fifteen regions.

Region $k^*_{0'}$: The cycle is $\theta'_0(k^*)$, and its throughput is $\tau'_0(k^*)$.

Region $k^*_i$: The cycle is $\theta_i(k^*)$, and its throughput is $\tau_i(k^*)$, $i = 0, \dots, 9$.

Region $k^*_{10}$: The cycle is $\theta_7(k^*)$ or $\theta_8(k^*)$, and its throughput is $\tau_7(k^*)$ or $\tau_8(k^*)$.

Region $k^*_{11}$: The cycle is $\theta_8(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_8(k^*)$ or $\tau_9(k^*)$.

Region $k^*_{12}$: The cycle is $\theta_7(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_7(k^*)$ or $\tau_9(k^*)$.

Region $k^*_{13}$: The cycle is $\theta_7(k^*)$, $\theta_8(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_7(k^*)$, $\tau_8(k^*)$ or $\tau_9(k^*)$.

Example 5 in the online Appendix illustrates the case of $2 > r > 1$. Note that the maximal throughput for Example 5 is $v_1 + v_2 = 3.8$. The current work content $(s_1, s_2, s_3)$ on each station in Example 5 does not realize the highest throughput this *seru* can achieve. Recall the remark in Section 2 that we can adjust the work content on the stations to generate the maximal throughput. We materialize this remark as Theorem 3, in which the conditions for achieving the maximal throughput for different initial states are given. We provide the details for the case $k = k^* - 1$ because layer $D(k^* - 1)$ is the most complex and dynamic layer. Let $\delta_1 = \frac{k+1-kr}{r+1}$, $\delta_2 = k(r-1)$, $\delta_3 = \frac{r(k+1-kr)}{r+1}$, and $\delta_4 = \frac{(2k+1)(r-1)}{r+1}$.

**<u>THEOREM 3</u>**. If $2 > r > 1$, a two-worker three-station rotating *seru* achieves its maximal throughput $v_1 + v_2$ as follows.

✧ If $a^{(0)} \in \{p_1, b_1\}$, the maximal throughput is achieved by either of

(1)$s_1 = \delta_1$ and $\delta_2 \le s_2 \le \delta_3$; (2)$s_2 = \delta_1$ and $\delta_2 \le s_1 \le \delta_1$; or (3)$s_3 = \delta_1$ and $\delta_4 \le s_1 \le \delta_1$.

✧ If $a^{(0)} \in \{p_2, b_2\}$, the maximal throughput is achieved by either of

(1)$s_2 = \delta_1$ and $\delta_2 \le s_3 \le \delta_3$; (2)$s_3 = \delta_1$ and $\delta_2 \le s_2 \le \delta_1$; or (3)$s_1 = \delta_1$ and $\delta_4 \le s_2 \le \delta_1$.

✧ If $a^{(0)} \in \{p_3, b_3\}$, the maximal throughput is achieved by either of

(1)$s_3 = \delta_1$ and $\delta_2 \le s_1 \le \delta_3$; (2)$s_1 = \delta_1$ and $\delta_2 \le s_3 \le \delta_1$; or (3)$s_2 = \delta_1$ and $\delta_4 \le s_1 \le \delta_1$.

Example 6 in the online Appendix illustrates Theorem 3. Note that the maximal throughput $v_1 + v_2 = 3.8$ is achieved in Example 6.

## 4. *m*-Station Two-worker Rotating *Serus* with Nonpreemptive Discrete Stations

The complexity of analyzing a rotating *seru* with *m* discrete stations and two workers increases significantly as the number of stations increases. However, the case of a three-station *seru* can be generalized to a rotating *seru* with *m* stations.

Passing is an inevitable aspect of rotating *serus* that use a rabbit-chasing mechanism. It is desirable for passing to always occur at a fixed station, as this allows for the implementation of appropriate measures to improve the performance of the *seru*. Workers should be trained to execute passing efficiently, as this reduces interference between workers and substantially improves productivity.

The designs for passing stations vary depending on the product being produced. However, a general principle is to allow faster and slower workers to work on their own items simultaneously at the passing station. For example, in a Japanese automobile component assembly factory in *Kyushu* (Matsuo, 2013), the assembly process has five discrete stations with the fifth station specifically designed to facilitate a high-velocity worker passing a low-velocity worker. The fifth station has additional space and tools to accommodate more than one worker. Another example is a factory of Itoki Corporation that uses a rotating *seru* to assemble components for office and healthcare facilities; the first station is the passing station, where workers can work simultaneously when a faster worker is passing a slower worker (Kakigi 2003).

For a large $m$-station rotating *seru*, sufficient conditions have been identified that allow all passings to occur on a single fixed station. A fixed passing station with specific work content can help a large system achieve its maximal throughput.

Let $r$ be the velocity ratio of two workers and $s_i$ $(i = 1,2, \dots, m)$ be the fixed station where passings occur. There is a cycle, denoted as $p_i \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_i$, where $k$ is the occurrence of $c_2$ during the time elapsed between two successive states. There are two exclusive cases: $r \geq 2$ and $2 > r > 1$.

### 4.1. $r \geq 2$

**LEMMA 8.** For a large $m$-station rotating *seru*, where $r \geq 2$, if the cycle is $p_i \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_i, i = 1,2, \dots, m$, then $k = 0$.

The above lemma is an application of Lemma 2. A sufficient condition for passings to occur at a fixed station is as follows.

**THEOREM 4.** For a large $m$-station rotating *seru*, where $r \geq 2$, with an initial state of $p_i$ or $b_i$, if $s_i \geq 1/(r+1)$, then passings will only occur at station $i$ $(i = 1,2, \dots, m)$; that is, we have the cycle $p_ic_1p_i$. When $s_i = 1/(r+1)$, the maximal throughput is obtained, that is $(r+1) \times v_2 = v_1 + v_2$.

Theorem 4 is a generalized extension of Theorem 2. Example 3 can also be used to explain Theorem 4. The next case, where $2 > r > 1$, is more complicated.

### 4.2. $2 > r > 1$

**LEMMA 9.** For a large $m$-station rotating *seru*, where $2 > r > 1$, if the cycle is $p_i \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_i$, $i = 1,2, \dots, m$, $k$ has the following two cases:

(A) If $s_i \geq 1/(r+1)$, then $k = 0$;

(B) If $s_i < 1/(r+1)$, then $1 \leq k \leq k^*$, where $k^* = \left\lceil \frac{m-1-r}{m(r-1)} \right\rceil$, a rounded-up integer.

Example 7 in the online Appendix illustrates Lemma 9.

Recalling again that $k^*$ is related to the number of completed items by the slower worker between two states $a^{(t)}$ and $a^{(t+1)}$, using the above lemma, a sufficient condition for passing to occur at a fixed station is as follows.

**THEOREM 5.** For a large $m$-station rotating *seru*, where $2 > r > 1$, with an initial state of $p_i$ or $b_i$, if the following conditions are satisfied, then passings will only occur at station $i$, $i = 1,2,\dots,m$; that is, we have the cycle $p_i \underbrace{c_1c_2 \dots c_1c_2}_{k} c_1p_i$, $k = 0,1,2,\dots,k^*$, where $k^* = \left\lceil \frac{m-1-r}{m(r-1)} \right\rceil$. There are two exclusive cases:

(A) $k = 0$. $s_i \geq 1/(r+1)$ (1)

When $s_i = 1/(r+1)$, the maximal throughput is obtained: $(r+1) \times v_2 = v_1 + v_2$.

(B) $k = 1,2,\dots,k^*$.

$$\begin{cases} k + r - kr - rs_j > rs_i + (r-1)\sum_{o=i+1}^{j-1} s_o \geq s_j, & j = i+1,\dots,m; \\ k + r - kr - rs_j > rs_i + (r-1)\sum_{o=i+1}^{m} s_o + (r-1)\sum_{o=1}^{j-1} s_o \geq s_j, & j = 1,\dots,i-1. \end{cases} \quad (2)$$

$$\begin{cases} k + 1 - kr \leq (r+1)s_i < k + r - kr, \\ (r+1)s_j < k + r - kr, & j = 1,\dots,m, j \neq i. \end{cases} \quad (3)$$

When $s_i = \frac{k+1-kr}{r+1}$, the maximal throughput is obtained: $(r+1) \times v_2 = v_1 + v_2$.

Example 8 in the online Appendix illustrates Theorem 5.

The managerial benefits from Theorems 4 and 5 are clear. Based on the different velocity ratios, $r$, a manager can adjust the work content distributed on stations to obtain the maximal throughput $v_1 + v_2$. With an initial state of $p_i$ or $b_i$, for the case of $r \geq 2$, if $s_i = 1/(r+1)$, or for the case of $2 > r > 1$, if $s_i = 1/(r+1)$ or $s_i = (k + 1 - kr)/(r+1)$, the faster worker passes the slower worker only on station $i$, and the maximal throughput is achieved.

Theorems 4 and 5 provide guidelines for designing a rotating *seru* with discrete stations in order to maximize throughput. In practice, for example, the managers at the Japanese automobile component assembly factory mentioned earlier use their experience and trial-and-error methods to design passing stations and distribute the work content among the stations. The results of Theorems 4 and 5 can assist managers in obtaining the optimal design for a rotating *seru*. To further illustrate the practical applications of our research, we provide two concrete examples of order picking and production lines.

Recall the order picking and production applications of rotating *serus* introduced in Section 1. Theorems 4 and 5 can be used to designate a passing station based on the workers' velocities in order to achieve the highest throughput. For example, consider two workers who have velocities of 2 and 1 ($r = 2$), respectively. For an order picking rotating *seru* whose task is to pick 9 different items, following Theorem 4, managers could create 7 stations. Each station would contain one bin (shelf). The final station

would stock 3 items and each of the other six stations would stock a single item. Thus, the work content (or the *density*, in the language of order picking literature; see Batt and Gallino (2019)) of the final station would be 3, and the work content of each of the other six stations would be 1. Then, the faster worker would always pass the slower worker at the final station, and the maximal throughput of 3 could be achieved.
Let us now consider a real-life production case, the product introduced in Chapter 4 of Cachon and Terwiesch (2012), a kick scooter produced by Novacruz. (See Cachon and Terwiesch's book for details.) The assembly process of this scooter requires 26 tasks, each of which has its own assembly time. A station can contain one or more tasks. If we use a rotating *seru* with two workers ($r = 2$) to assemble this product, the possible number of stations for the rotating *seru* can range from 1 to 26. This means that the work content of each station can be adjusted based on the number of tasks it contains. The work content of this product is 1,890 seconds. By applying Theorem 4, we can design the first station to contain tasks 1-10 with a work content of 623 seconds. As a result, the faster worker will always pass the slower worker at the first station. Additionally, the first task (preparing the cable) can be specially designed to allow the faster worker to pass the slower worker smoothly.

## 5. $m$-Station $n$-Worker Rotating *Serus* with Nonpreemptive Discrete Stations

The problem discussed in Section 4 is a two-dimensional (two workers) problem. In Section 4, we saw that the complexity of analyzing a rotating *seru* with $m$ discrete stations and two workers increases rapidly with the number of stations. In this section, we show that the analysis of a higher dimensional problem—a rotating *seru* with $m$ discrete stations and $n$ ($n \geq 3$) workers—is almost impossible because such a system tends to be chaotic. The existence of chaos in a dynamical system implies that no one can precisely predict the future result of the system function $f$ (Zhang, 2006). One way to prove the chaos of a system is to show that the system function has periodic orbits of every period. For an $m$-station $n$-worker rotating *seru*, this implies that the *seru* has period-1, period-2, …, period-$m$ orbits. An amazing and important theorem given by Li and Yorke (1975) is that if $f$ has a periodic orbit of period three, then $f$ has periodic orbits of every period. Following the Li-Yorke Theorem, Example 9 in the online Appendix illustrates a period-3 orbit, which means that higher dimensional rotating *serus* (e.g., three workers and four stations) tend to be chaotic.

Since it is almost impossible to predict a chaotic system, we hereafter conduct simulated experiments to identify properties of higher dimensional rotating *serus*.

### 5.1. Simulated experiments

Let $v_j$ ($j = 1,2,\dots,n$) denote the worker velocity of worker $j$, where $v_1 > v_2 > \cdots > v_n$. Define $\delta = v_1/v_n$ as the velocity ratio of workers. We have $v_j = \Big(1 -$

$\left.\frac{\delta-1}{\delta(n-1)} \times (j-1)\right) \times v_1$, for $j = 1,2,\ldots,n$. We assume that faster workers have higher priority. That is, when multiple workers are blocked at the start of a certain station at the same time, they are allowed to pass that station based on their priorities, from highest to lowest. We evaluate the average throughput of workers under the impacts of velocity ratios, worker numbers, station numbers, and varied work velocities. In experiments (1)-(3), after running 500 warm-up items, 2000 items are run to compute the average throughput of workers.

**Experiment (1): Impacts of velocity ratios and worker numbers**

In this experiment, three factors that impact throughput are considered: velocity ratio $\delta$, worker number $n$, and station number $m$. Figure 8(a) illustrates the effects of changes in the number of workers and the velocity ratio. We set the average velocity of all workers as $\bar{v} = 0.5$. We vary the velocity ratio $\delta$ from 1.2 to 2.8 (as shown by the various curves in Figure 8(a)). The change in the number of workers is represented on the horizontal axis, with values ranging from 3 to 9 ($n = 3,\ldots,9$).

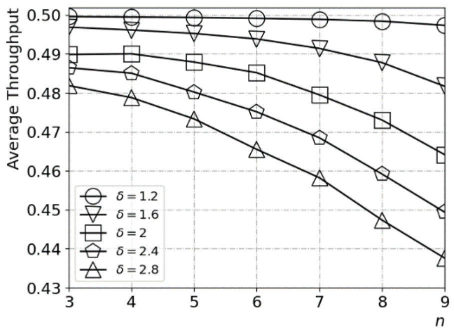
The vertical axis in Figure 8(a) represents the grand average throughput, which is defined as the average of the average throughputs of different rotating *serus*. The calculation for the grand average throughput is as follows: different station numbers ($m = 10,11,\ldots,20$) are tested for the rotating *serus*. The length of each station is $s_i = 1/m$ ($i = 1,2,\ldots.m$), where $m$ is the number of stations within a rotating *seru*. For each combination of worker number and velocity ratio (e.g., $n = 9$ and $\delta = 1.2$), 11 rotating *serus* are tested, differentiated by their station numbers ($m = 10,11,\ldots,20$). The average throughput of workers for each *seru* is calculated, resulting in 11 average throughputs for each combination of $n$ and $\delta$. The grand average throughput is the average of these 11 average throughputs. For example, the point at the top right of Figure 8(a) represents the grand average throughput of the combination $n = 9$ and $\delta = 1.2$.

Figure 8(a) illustrates that as the number of workers, $n$, increases, the throughput decreases. For instance, when $\delta = 1.6$ and $n = 4$, the throughput is 0.496. However, when $\delta = 1.6$ but $n = 6$, the throughput decreases to 0.494. The reason for this is that as $n$ increases, there are more possibilities for blocking and passing to occur. Similarly, as the velocity ratio, $\delta$, increases, the throughput decreases. For example, when $n = 5$ and $\delta = 1.2$, the throughput is 0.499. However, when $n = 5$ and $\delta = 2.0$, the throughput decreases to 0.488. The reason for this is that as $\delta$ increases, faster workers are more likely to frequently pass slower workers.

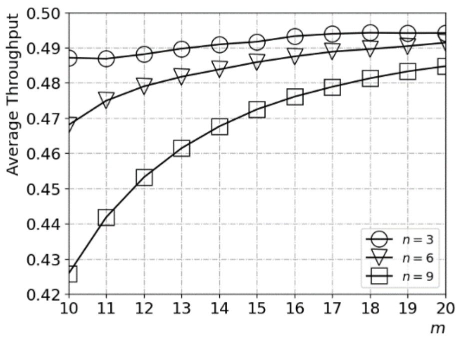
**Experiment (2): The impact of station numbers**

In this experiment, the impact of changes in the station number $m$ is tested. The velocity ratio is varied from 1.1 to 2.9 ($\delta = 1.1, 1.2, \ldots, 2.9$), while other factors ($m$, $\bar{v}$, $n$, $s_i$) remain the same as in Experiment (1). The results are presented in Figure 8(b), where the horizontal axis represents the number of stations $m$ and the vertical axis shows the average throughput of workers. For example, the point at the top right of Figure 8(b) represents the average throughput for the combination of $n = 3$ and $m = 20$.

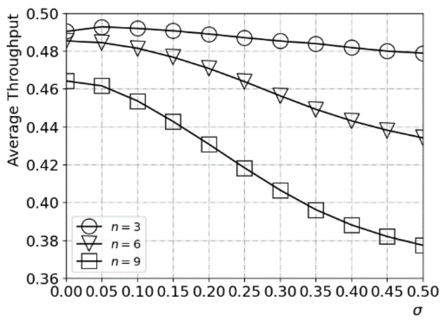
The result of this experiment demonstrates that the throughput increases as the number of stations, $m$, increases. For instance, when $n = 6$ and $m = 13$, the throughput is 0.483. However, when $n = 6$ and $m = 18$, the throughput increases to 0.490. This is because as $m$ increases, workers have more possibilities to work at different stations, leading to fewer instances of blocking and passing.




(a) velocity ratios and worker numbers




(b) station numbers




(c) the variability of velocities

Figure 8. Impacts of different factors

**Experiment (3): The impact of the variability of work velocities**

As discussed in Section 1, the problem being studied in this paper falls under the category of constant velocity and nonpreemptive, as indicated in the upper right quadrant of Matrix 1. Problems that are nonpreemptive are considered more challenging than those that are preemptive, and problems with varied velocity are considered more difficult than those with constant velocity. To the best of our knowledge, no studies have been published in the category of varied velocity and nonpreemptive, which poses the most challenging research questions.

This experiment examines the effects of velocity variability. We assume that when worker $j$ is working on her/his $t$-th item, her/his velocity is $\tilde{v}_j^t = v_j/(1 + \varepsilon_j^t)$, where $v_j$ is constant and $\varepsilon_j^t$ is an independent and identically distributed random variable, for $j = 1,2,\ldots,n$ and $t = 1,2,3,\ldots$. Under this assumption, a worker has a different velocity each time she/he initiates an item. The results reported are for the case where each $\varepsilon_i^t$ follows a normal distribution $\mathcal{N}(0,\sigma^2)$, where $\sigma$ is the standard deviation.

The result of this experiment is shown in Figure 8(c). The horizontal axis is $\sigma$, which varies from 0 to 0.5 with a stepwise increase of 0.05. The vertical axis displays the grand average throughput, which is calculated in the same way as in Experiment (1). We set $\delta = 2$. All other factors ($m$, $\bar{v}$, $n$, $s_i$) are kept the same as in Experiment (1).

From Figure 8(c), it is clear that the throughput decreases as the velocity variability $\sigma$ increases. For example, when $\sigma = 0.15$ and $\sigma = 0.40$, if $n = 3$, the throughputs are 0.491 and 0.482 respectively, representing a drop of 1.8%. Similarly, when $\sigma = 0.15$ and $\sigma = 0.40$, if $n = 9$, the throughputs are 0.443 and 0.388, respectively, representing a drop of 12.4%. The results in Figure 8(c) indicate that a smaller number of workers, $n$, can better handle the impact of velocity variability. The is because as $n$ decreases, there are fewer instances of blocking and passing among workers.

## 6. Comparisons and Managerial Insights

In this section, we compare rotating *serus* to cellular bucket brigades, and based on the results from previous sections, we summarize the managerial insights of this study.

### 6.1. Comparisons with cellular bucket brigades

We examine a cellular bucket brigade with $m$ nonpreemptive discrete stations and two workers. Figure 9 depicts the movement of the two workers within the cellular bucket brigade. Worker 1 carries an item until he reaches the end of station $i$. He then exchanges the item with worker 2, and continues working on station $j+1$ until the item he receives from worker 2 is completed. Then, he initiates work on a new item. Similarly, worker 2 carries an item until he reaches the end of station $j$. He then exchanges the item with worker 1, and continues on station $i+1$. That is, worker 1 works on stations $s_1 \sim s_i$ and $s_{j+1} \sim s_m$, while worker 2 works on $s_{i+1} \sim s_j$, where $0 \leq i < j \leq m$. It is important to note that all assumptions used in this comparison are identical to those utilized in the study by Lim and Wu (2014).

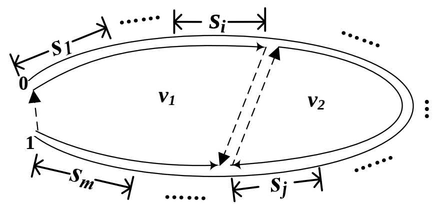



Figure 9. A cellular bucket brigade with discrete and nonpreemptive stations.

We consider the scenario in which both a rotating *seru* and a cellular bucket brigade have achieved their highest throughputs, represented by $v_1 + v_2$, where $v_1$ and $v_2$

are the velocities of workers 1 and 2, respectively. However, the performance of a production line is impacted by various internal and external factors, such as hand-offs, the tradeoff between specialization and diversification in productivity, and workflow and layout design, among others. In this comparison, we focus on the first two factors—hand-offs and the tradeoff between specialization and diversification—as these are deemed the most relevant. Hand-offs refer to the exchange of items between two workers, which is a necessary step in both cellular bucket brigades and rotating *serus*. In addition, the tradeoff between specialization and diversification must be taken into account as workers in cellular bucket brigades have a limited set of tasks to perform, while workers in rotating *serus* perform all tasks on all stations. This may result in a decrease in productivity as workers may not have the same level of proficiency when performing a wider range of tasks and may take longer to complete each task.

Suppose that the rotating *seru* operates on a behavior cycle $p_i \underbrace{c_1 c_2 \ldots c_1 c_2}_{k} c_1 p_i$, $k \in \{0,1,2,\ldots,k^*\}$, where $k$ and $k^*$ are defined as in Section 4. The assembly of an item in a bucket brigade requires one hand-off, compared to $1/(2k+1)$ hand-offs in a rotating *seru*. Let $h$ be the hand-off time required by two workers to exchange items. The impact of the tradeoff between specialization and diversification is described by a velocity scaling factor $\mu \in (0,1)$. $v_i$ and $\mu \times v_i$ are the velocities of worker $i$ in the cellular bucket brigade and rotating *seru*, respectively.

The average throughput equals the maximum possible throughput $v_1 + v_2$ offset by the waste time in the hand-offs. The throughput of a cellular bucket brigade is

$$\tau_c = \frac{v_1+v_2}{1+h(v_1+v_2)} \tag{4}$$

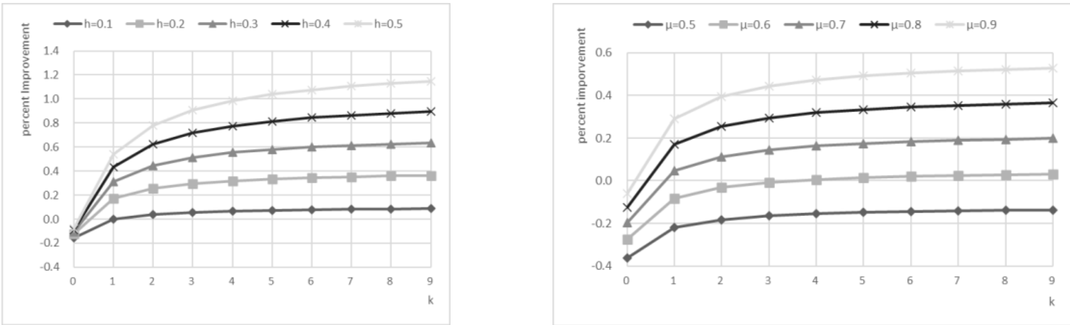
Correspondingly, the throughput of a rotating *seru* is offset to

$$\tau_r = \frac{v_1+v_2}{1/\mu+h(v_1+v_2)/(2k+1)} \tag{5}$$

Define the *percent improvement* in throughput achieved by the rotating *seru* in comparison to the cellular bucket brigade as $[(\tau_r - \tau_c)/\tau_c] \times 100\%$. Figure 10 illustrates that the percent improvement increases as $k$ increases. In Figure 10(a), we set $\mu = 0.8$ and vary $h$ from 0.1 to 0.5. In Figure 10(b), we set $h = 0.2$ and vary $\mu$ from 0.5 to 0.9. For both Figures 10(a) and 10(b), we set $v_1 = 2.0$ and $v_2 = 1.8$.

The results in Figure 10 indicate that rotating *serus* have a significant advantage over cellular bucket brigades when *k* is large. At *k*=8, with *h*=0.5 and $\mu$=0.8, the improvement in throughput is as high as 113%; and when *k*=8, *h*=0.2, and $\mu$=0.9, the improvement is as high as 52%. However, as *k* gets smaller, the percent improvement decreases and can even become negative in some cases. This is because the hand-off times in a rotating *seru* are affected by the value of *k*.

It is important to note that $k$ is bounded by $k^*$, which increases with an increase in the number of stations $m$ and a decrease in the velocity ratio of workers $r$. This observation suggests that a large number of stations and a small difference in worker velocities favors the use of rotating *serus*. This is because, by properly adjusting the work content distribution on stations, rotating *serus* have a greater possibility of wasting less time on hand-offs as $k^*$ becomes larger.

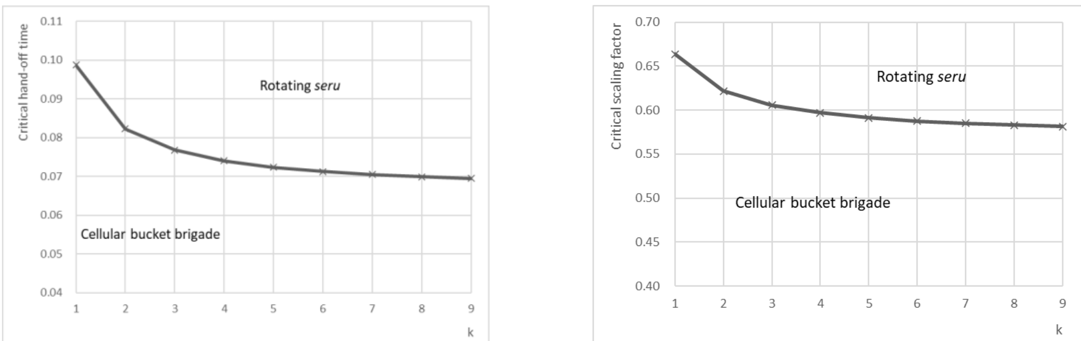



(a) $\mu = 0.8$ (b) $h = 0.2$

Figure 10. Throughput improvement by the rotating *seru*.

The hand-off time $h$ and the scaling factor $\mu$ also play a significant role in Figure 10. When $k = 0$, the rotating *seru* never outperforms the cellular bucket brigade due to $\mu < 1$. When $k > 0$, the rotating *seru* outperforms the cellular bucket brigade if either of the following two conditions is satisfied. First, given a scaling factor μ, the hand-off time $h$ is larger than the critical hand-off time $h^* = \frac{(1-\mu)(2k+1)}{2k\mu(v_1+v_2)}$. Second, given a hand-off time $h$, the scaling factor μ is larger than the critical scale factor $\mu^* = \frac{2k+1}{2k+1+2kh(v_1+v_2)}$. These two critical values can be obtained from Equations (4) and (5).

The curves in Figure 11 represent the critical hand-off times and the critical scale factors. In Figure 11(a), the rotating *seru* (cellular bucket brigade) is favored when the hand-off is located in the region above (below) the curve. For example, when $k = 2$, the rotating *seru* is the preferred choice if the hand-off time is higher than 0.082. Similarly, in Figure 11(b), the rotating *seru* (cellular bucket brigade) is preferred when the scale factor is located in the region above (below) the curve. For example, when $k = 2$, the rotating *seru* is preferred if the scaling factor is greater than 0.622.



(a) The critical hand-off time, $\mu = 0.8$. (b) The critical scaling factor, $h = 0.2$.

Figure 11. The critical hand-off time and velocity scale.

**6.2. Managerial insights**

The comparison in Section 6.1 reveals that neither the rotating *seru* nor the cellular bucket brigade is a one-size-fits-all production mode. The performance of a production line can vary in terms of hand-off times and velocity scale factors. The choice between the rotating *seru* and the cellular bucket brigade will depend on the specific production context and requirements. The rotating *seru* can provide a better solution in cases where $h$, $\mu$ and $k$ are high, while the cellular bucket brigade is a better choice in other cases. The important thing is to consider both options and determine which one will provide the best performance for the specific production scenario. This observation leads to the following managerial insight.

***MI1***: *There is no universally best assembly system. The rotating seru provides an alternative or option for existing production systems.*

Based on MI1, we identify under which conditions a rotating *seru* is the best choice. The use of *serus* in production has become increasingly popular due to the need to adapt to volatile markets characterized by frequently changing product models, short product life cycles, and fluctuating demand. Rotating *serus* are particular suitable for handling fluctuations in production volume. This is achieved by adjusting the number of workers within the *seru* production system. For example, during a demand surge, to increase production capacity, additional workers can be added into exist *yatais* (recall that a *yatai* is a rotating *seru* with only one worker), converting them into rotating *serus*, and then returning them to their original form after the surge. This adaptability is a key advantage of rotating *serus*, separating them from traditional assembly lines and other production systems. This approach helps companies to be more nimble and responsive to market changes, making them more agile and resilient in their operations.

However, based on the results of this study, it was found that adding workers to *yatais* without optimizing the rotating *seru* designs can increase the human capacity (i.e., the number of workers), but may also lead to a decrease in the overall performance of the newly created rotating *serus*. For instance, consider a scenario where a worker with velocity 1 is added into a 3-station *yatai*, where the other worker's velocity is 2. Assume that the work content at the 3 stations is 8/10, 1/10, and 1/10, respectively. If the first station is designated as the passing station, using Theorem 2 in Section 3.1, the throughput of the rotating *seru* is calculated to be 10/9. However, the original throughput of the *yatai* was 2; in this case the addition of another worker resulted in a 44% decrease in the throughput. This leads to our second managerial insight.

***MI2**: To improve the throughput of a seru system, the addition of workers must be approached strategically and scientifically. Otherwise, it may have a detrimental effect on the performance of the seru system.*

The significance of MI2 lies in highlighting the potential obstacles and unintended consequences that may arise from the improper use of rotating *serus*. The objective of this research is to address these issues by offering optimized rotating *seru* design methods through Theorems 2, 3, 4, and 5.

The significance of Theorems 2, 3, 4, and 5 lies in their potential to provide a simple and manageable approach for managers to optimize rotating *seru* design in a volatile environment. As demonstrated by management scholars (Davis et al., 2009; Bingham and Eisenhardt 2011), simple rules are effective in providing a straightforward method for navigating and controlling complex organizations. Unlike mathematical models and software, simple rules are easy to understand and implement, enabling managers to have greater control over complex organizations without relying on complex technology. A good example of a successful simple rule in practice is the *Kanban* system used in the Toyota production system, which is widely recognized as an efficient method for managing a complex manufacturing process.

Theorems 2 and 4 in this study are simple, indicating that setting the work content of a station to $1/(r+1)$ results in the highest throughput when $r \geq 2$, meaning that the higher-speed worker is at least twice as fast as the slower worker. On the other hand, when $1 < r < 2$, Theorems 3 and 5 are not as simple, requiring the work content to be set to $(k+1-kr)/(r+1)$ to achieve the highest throughput. However, when closely examining the equation, it becomes clear that as $r$ approaches 1, $(k+1-kr)/(r+1)$ approaches the simple value of $1/(r+1)$, indicating that the workers have similar velocities. This managerial insight is crucial for implementating a rotating *seru*.

***MI3**: To maximize throughput, a worker should be added who is either at least twice as fast or at most half as fast as the current worker in the yatai, or has a similar velocity. The work content should then be set using the simple value* $1/(r+1)$.

## 7. Conclusions

*Seru* production systems have gained significant attention from researchers and practitioners due to their high responsiveness capabilities. de Treville et al. (2017) and Yin et al. (2017) have emphasized their usefulness for countries with high-cost, high-skill labor. Lewis (2020) stated in a comprehensive overview of operations management (OM) research that *seru* production is an extension of lean production that can quickly respond to volatile market demand. A comparison between *serus* and other manufacturing systems can be found in Yin et al. (2018). In current competitive business environments (Xu 2021; Park 2022), innovative process innovations are required (Matsuno et al. 2021; Zhang et al. 2021; Wang and Li 2022; Wang et al., 2022), and the *seru* production system has the potential to meet this demand. The fundamental management and control principles of *seru* production systems have been outlined by Stecke et al. (2012), Liu et al. (2014), and Yin et al. (2017).

The use of rotating *serus* in Asian electronics manufacturing companies for assembly systems has been prevalent, yet theories for their implementation have been limited. This study analyzed the mechanisms of a rotating *seru* with discrete and nonpreemptive stations through the use of theories from dynamical systems. Within such a *seru*, only one worker is allowed to work on a station at any time, and her/his work cannot be interrupted within a station. Rotating rules are introduced and summarized. The goal was to identify conditions that would lead to maximum throughput. The cycles and throughputs of three-station two-worker rotating *serus* were analyzed and determined, and the results were extended to larger systems. The findings, presented in Theorems 4 and 5, provide guidelines for designing a rotating *seru* with discrete stations to maximize throughput, making them a valuable resource for managers in floor shops. This study's results can help managers achieve the optimal design for maximum throughput without relying on trial-and-error methods. The chaotic characteristics of rotating *serus* were also demonstrated, and simulations were run to assess the impact of various factors on throughput.

Previous studies have viewed rotating *serus* as a black box, as seen in the works of Gai et al. (2022), Liu et al. (2022), Zhan et al. (2022), Zhang et al. (2022a, 2022b, 2022c, 2022d), and Li (2023). This study is the first to open the black box by analyzing the inside mechanisms of rotating *serus*. Factors affecting *serus*, such as random work velocities at stations, different walking speeds, and others, are relevant and intriguing topics for future research. In addition, a comparison among the three types of *serus*—rotating *serus*, divisional *serus* (staffed by partially cross-trained workers with tasks divided into different sections), and *yatais*—would be an interesting area for analysis.

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## Online Electronic Appendix

In this online electronic appendix, Section A.1 includes proofs for all theorems and lemmas, Section A.2 includes all illustrative examples mentioned in the paper.

### A.1. Proofs for theorems and lemmas

✧ **Proof of THEOREM 1.**

First, we prove the existence of a cycle. Given arbitrary initial positions of $x_1$ and $x_2$, we can find the initial state $a^{(0)}$, and we have an orbit $\{a^{(0)}, a^{(1)}, \dots, a^{(t)}, \dots\}$ of passing or blocking behaviors under the function $a^{(t+1)} = f(a^{(t)})$. Correspondingly, we have the orbit $\{i^{(0)}, i^{(1)}, \dots, i^{(t)}, \dots\}$ of station indices, where $i^{(t)}$ is the index of the station where $a^{(t)}$ occurs. We continue iterating through these two orbits synchronously until we reach the first behavior $a^{(w)}$ with which we have $a^{(w)} = a^{(u)}$ and $i^{(w)} = i^{(u)}$, $u \in \{0,1,\dots,w-1\}$. Since the number of stations is limited and the velocity of the first worker $v_1$ is faster than the velocity of the second worker $v_2$, we can always find these two identical behaviors $a^{(u)}$ and $a^{(w)}$. By the rotating *seru* rules of Section 2.1, because $i^{(w)} = i^{(u)}$, the positions of the two workers after behaviors $a^{(w)}$ and $a^{(u)}$ are the same. In this way, a cycle $\{a^{(u)}, a^{(u+1)}, \dots, a^{(w)}\}$ is constructed. We call it a period-*n* cycle, where $n = w - u$.

Next, we prove the uniqueness of the cycle $\{a^{(u)}, a^{(u+1)}, \dots, a^{(w)}\}$ for any combination of $(x_1, x_2, v_1, v_2)$, where $a^{(w)} = a^{(u)}$. Assume that this is not true and there exist two distinct cycles beginning with an initial state $a^{(0)}$. This means that there exists at least one passing or blocking behavior in the cycle $\{a^{(u)}, a^{(u+1)}, \dots, a^{(w)}\}$, which has two different successors (i.e., two different positions of worker 1 and two different positions of worker 2). However, for the two-worker and *m*-station rotating *seru*, the successor of any passing or blocking behavior is unique, which leads to a contradiction. □

✧ **Proof of LEMMA 1.**

(1). If $c_2$ does not occur, there is a single instance of $c_1$ between $a^{(t)}$ and $a^{(t+1)}$ (i.e., $a^{(t)} c_1 a^{(t+1)}$). If there were two or more instances of $c_1$ between $a^{(t)}$ and $a^{(t+1)}$, there would have to be a passing behavior $c_1 p c_1$ between each consecutive $c_1$, which would result in a contradiction.

Now, if $c_2$ does occur between $a^{(t)}$ and $a^{(t+1)}$, let the positions of workers 1 and 2 be $x_1$ and $x_2$, respectively. Regardless of the behavior of $a^{(t)}$ (i.e., passing or blocking), after $a^{(t)}$, the relationship between the workers' positions will be $x_2 < x_1$, meaning that worker 2 is closer to the entrance and worker 1 is closer to the exit. Therefore, if the behavior after $a^{(t)}$ is either $c_1$ or $c_2$, it must be $c_1$, not $c_2$.

(2). If $c_1$ and $c_2$ do not occur alternatively, there are two cases to consider.

Case 1. Two or more $c_1$ occur successively. Before $c_1$, the relationship between the workers' positions must be $x_2 < x_1$. After $c_1$, the relationship changes to $x_1 < x_2$. Before the next $c_1$, the relationship must be $x_2 < x_1$ again. Only two behaviors can change $x_1 < x_2$ to $x_2 < x_1$: a $c_2$ or worker 1 passing worker 2. Because the behaviors are in a repeating pattern, it must be $c_2$.

Case 2. Two or more $c_2$ occur successively. The proof is similar to case 1.

(3). If $a^{(t+1)} \in P$, before $a^{(t+1)}$, the relationship between the workers' positions is $x_1 < x_2$. Therefore, if the behavior before $a^{(t+1)}$ is circling, it must be $c_1$. From property (1), the behavior after $a^{(t)}$ is $c_1$; Form property (2), $c_1$ and $c_2$ occur alternatively. So, the number of occurrences of $c_1$ is one more than that of $c_2$.

(4). If $a^{(t+1)} \in B$, before $a^{(t+1)}$, the relationship between the workers' positions is $x_2 < x_1$. Therefore, if the behavior before $a^{(t+1)}$ is circling, it must be $c_2$. From property (1), the behavior after $a^{(t)}$ is $c_1$; Form property (2), $c_1$ and $c_2$ occur alternatively. So, the number of occurrences of $c_1$ is equal to that of $c_2$. □

✧ **Proof of LEMMA 2.**

(1) We set $a^{(t)} \in \{p_i, b_i\}$ and $a^{(t+1)} = p_j$, where $i, j = 1,2, \ldots, m$.

From (3) of Lemma 1, we have $a^{(t)}(c_1c_2)^k c_1 a^{(t+1)}$. We show $k \leq \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil + 1$ as follows.

There are two exclusive cases: (A) $r \geq 2$ and (B) $2 > r > 1$.

(A) $r \geq 2$

Since $r \geq 2$, it follows that $\frac{2}{v_1} \leq \frac{1}{v_2}$. Therefore, the next iterate (or state, we use them interchangeably hereafter) will occur before worker 2 can reach the start of station $i$ again. Thus, we have $k \leq 1$.

(B) $2 > r > 1$

Suppose that $k > \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil + 1$. Then, $k \geq \frac{m-1-r}{m(r-1)} + 2$. This implies that $\frac{k-2+1/m}{v_2} \geq \frac{k-1-1/m}{v_1}$. We have two exclusive cases: (a) $s_i \geq 1/m$ and (b) $s_i < 1/m$.

(a) $s_i \geq 1/m$

If $s_i \geq \frac{1}{m}$, then we have $\frac{k-2+s_i}{v_2} \geq \frac{k-2+1/m}{v_2} \geq \frac{k-1-1/m}{v_1} \geq \frac{k-1-s_i}{v_1}$. This implies that the next iterate will occur before worker 2 completes $k-1$ items and then reaches the end of station $i$. Thus, $b_i(c_1c_2)^k c_1 p_j$ or $p_i(c_1c_2)^k c_1 p_j$ cannot be constructed.

(b) $s_i < 1/m$

Assume that for any station $j'$, $s_{j'} < \frac{1}{m}$. We have $\sum_{j'=1}^{m} s_{j'} < 1$. Therefore, if $s_i < 1/m$, there exists at least one station $j'$ $(j' \neq i)$, $s_{j'} \geq 1/m$. Since $s_{j'} \geq 1/m$, we have $\frac{k-2+s_{j'}}{v_2} \geq \frac{k-2+1/m}{v_2} \geq \frac{k-1-1/m}{v_1} \geq \frac{k-1-s_{j'}}{v_1}$. There are two exclusive cases: (i) $i < j'$ and (ii) $i \geq j'$.

(i) $i < j'$

The time for worker 1 to complete $k-1$ items and reach the start of station $j'$ is $t_1 = \frac{s_{i+1}+\cdots+s_m+k-2+s_1+\cdots+s_{j'-1}}{v_1} = \frac{k-1+s_{i+1}+\cdots+s_{j'-1}}{v_1}$. The time for worker 2 to complete $k-2$ items and reach the end of station $j'$ is $t_2 = \frac{s_i+\cdots+s_m+k-3+s_1+\cdots+s_{j'}}{v_2} = \frac{k-2+s_i+\cdots+s_{j'}}{v_2}$. Combined with $\frac{k-2+s_{j'}}{v_2} \geq \frac{k-1-s_{j'}}{v_1}$, we have $t_2 \geq t_1 + \frac{s_i+\cdots+s_{j'-1}}{v_2} - \frac{s_{i+1}+\cdots+s_{j'}}{v_1}$.

If $\frac{s_i+\cdots+s_{j'-1}}{v_2} < \frac{s_{i+1}+\cdots+s_{j'}}{v_1}$, worker 2 can reach the start of station $j'$ before worker 1 finishes his work on station $j'$. Therefore, we have $a^{(t+1)} = b_{j'}$, and the behaviors $b_i(c_1c_2)^k c_1 p_j$ or $p_i(c_1c_2)^k c_1 p_j$ cannot be constructed.

If $\frac{s_i+\cdots+s_{j'-1}}{v_2} \geq \frac{s_{i+1}+\cdots+s_{j'}}{v_1}$, we have $t_2 \geq t_1$. The next iterate will occur before worker 1 completes $k-1$ items and reaches the start of station $j'$. Also, the behaviors $b_i(c_1c_2)^k c_1 p_j$ or $p_i(c_1c_2)^k c_1 p_j$ cannot be constructed.

(ii) $i \geq j'$

The time for worker 1 to complete $k$ items and reach the start of station $j'$ is $t_1 = \frac{s_{i+1}+\cdots+s_m+k-1+s_1+\cdots+s_{j'-1}}{v_1} = \frac{k-(s_{j'}+\cdots+s_i)}{v_1}$. The time for worker 2 to complete $k-1$ items and reach the end of station $j'$ is $t_2 = \frac{s_i+\cdots+s_m+k-2+s_1+\cdots+s_{j'}}{v_2} = \frac{k-1-(s_{j'+1}+\cdots+s_{i-1})}{v_2}$. Combined with $\frac{k-2+s_{j'}}{v_2} \geq \frac{k-1-s_{j'}}{v_1}$, we have $t_2 \geq t_1 + \frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j'-1})}{v_2} - \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_{j'})}{v_1}$.

If $\frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j'-1})}{v_2} < \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_{j'})}{v_1}$, worker 2 can reach the start of station $j'$ before worker 1 finishes his work on station $j'$. So, we have $a^{(t+1)} = b_{j'}$, the behaviors $b_i(c_1c_2)^k c_1 p_j$ or $p_i(c_1c_2)^k c_1 p_j$

cannot be constructed.

If $\frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j'-1})}{v_2} \geq \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_{j'})}{v_1}$, we have $t_2 \geq t_1$. The next iterate will occur before worker 1 completes $k$ items and reaches the start of station $j'$. Also, the behaviors $b_i(c_1c_2)^k c_1 p_j$ or $p_i(c_1c_2)^k c_1 p_j$ cannot be constructed.

Taken together Case (A) and Case (B), under the assumption of $k > \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil + 1$, the behaviors $b_i(c_1c_2)^k p_j$ or $p_i(c_1c_2)^k p_j$ cannot be constructed. So, we can conclude that if $a^{(t+1)} \in P$, we have $a^{(t)}(c_1c_2)^k c_1 a^{(t+1)}$ and $k \leq \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil + 1$.

(2) We set $a^{(t)} \in \{p_i, b_i\}$ and $a^{(t+1)} = b_j$, where $i, j = 1,2,\ldots,m$.

From (4) of Lemma 1, we have $a^{(t)}(c_1c_2)^k a^{(t+1)}$. We show $k \leq 1$ as follows.

The time for worker 1 to complete $k$ items and reach the end of station $j$ is $t_1 = \frac{(s_{i+1}+\cdots+s_m)+k-1+(s_1+\cdots+s_j)}{v_1}$. The time for worker 2 to complete $k$ items and reach the start of station $j$ is $t_2 = \frac{(s_i+\cdots+s_m)+k-1+(s_1+\cdots+s_{j-1})}{v_2}$.

There are two cases: (A) $i < j$ and (B) $i \geq j$.

(A) If $i < j$, then $k = 0$.

Suppose that $k > 0$.

We have $t_1 - t_2 = k\left(\frac{1}{v_1} - \frac{1}{v_2}\right) + \frac{s_{i+1}+\cdots+s_j}{v_1} - \frac{s_i+\cdots+s_{j-1}}{v_2}$.

If $\frac{s_i+\cdots+s_{j-1}}{v_2} > \frac{s_{i+1}+\cdots+s_j}{v_1}$, we have $t_1 < t_2$, since $\frac{1}{v_1} - \frac{1}{v_2} < 0$ and $k > 0$. Thus, we have $a^{(t+1)} = b_{j'}$, $j' \neq j$.

If $\frac{s_i+\cdots+s_{j-1}}{v_2} < \frac{s_{i+1}+\cdots+s_j}{v_1}$, we have $k = 0$, which contradicts with $k > 0$.

(B) If $i \geq j$, then $k = 1$.

Suppose that $k > 1$.

We have $t_1 - t_2 = (k-1)\left(\frac{1}{v_1} - \frac{1}{v_2}\right) + \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1} - \frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2}$.

If $\frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1} < \frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2}$, we have $t_1 < t_2$, since $\frac{1}{v_1} - \frac{1}{v_2} < 0$ and $k > 1$. Thus, we have $a^{(t+1)} = b_{j'}$, $j' \neq j$.

If $\frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1} > \frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2}$, we have $k = 1$, which contradicts with $k > 1$.

Taken together Case (a) and Case (b), under the assumption of $k > 1$, the behaviors $p_i(c_1c_2)^k b_j$ or $p_i(c_1c_2)^k b_j$ cannot be constructed. So, we can conclude that if $a^{(t+1)} \in B$, we have $a^{(t)}(c_1c_2)^k a^{(t+1)}$ and $k \le 1$. □

✧ **Proof of LEMMA 3.**

We construct the function $f$ for the following three cases separately: (A) $a^{(t)} \in \{p_1, b_1\}$, (B) $a^{(t)} \in \{p_2, b_2\}$, and (C) $a^{(t)} \in \{p_3, b_3\}$. We determine the next iterate $a^{(t+1)}$ by considering all possible combinations of work content distributions on stations.

(A) $a^{(t)} \in \{p_1, b_1\}$

The positions where workers 1 and 2 are located immediately after $p_1$ or $b_1$ are the end and the start of station 1, respectively. Since $r \ge 2 \Longrightarrow \frac{2}{v_1} \le \frac{1}{v_2}$, the next iterate will occur before worker 2 can complete an item and reach the start of station 1. There are five subcases, (a)-(e), with respect to the different work content distributions.

(a) $a^{(t+1)} = p_1$ if $s_1 \ge 1/(r+1)$.

If $s_1 \ge \frac{1}{r+1} \Longleftrightarrow \frac{s_2+s_3}{v_1} \le \frac{s_1}{v_2}$, worker 1 can complete an item and return to the start of station 1 before worker 2 reaches the end of station 1. Thus, $p_1$ will occur.

(b) $a^{(t+1)} = b_2$ if $rs_1 < s_2$.

If $rs_1 < s_2$ and $s_2 < 1 - s_1$, we have $s_1 < \frac{1}{r+1} \Longrightarrow \frac{s_1}{v_2} < \frac{1-s_1}{v_1}$. Thus, we can guarantee that $p_1$ will not occur.

If $rs_1 < s_2 \Longleftrightarrow \frac{s_2}{v_1} > \frac{s_1}{v_2}$, worker 2 can reach the start of station 2 before worker 1 finishes his work on station 2. Thus, $b_2$ will occur.

(c) $a^{(t+1)} = b_3$ if $rs_1 \ge s_2$ and $rs_1 + (r-1)s_2 < s_3$.

If $rs_1 + (r-1)s_2 < s_3$ and $s_3 = 1 - s_1 - s_2$, we have $(r+1)s_1 < 1 - rs_2 < 1 \Longrightarrow s_1 < \frac{1}{r+1}$. Thus, we can guarantee that $p_1$ will not occur.

If $rs_1 \ge s_2$, we can guarantee that $b_2$ will not occur.

If $rs_1 + (r-1)s_2 < s_3 \Longleftrightarrow \frac{s_1+s_2}{v_2} < \frac{s_2+s_3}{v_1}$, worker 2 can reach the start of station 3 before worker 1 finishes his work on station 3. Thus, $b_3$ will occur.

(d) $a^{(t+1)} = p_2$ if $rs_1 \ge s_2$, $rs_1 + (r-1)s_2 \ge s_3$, $s_1 < 1/(r+1)$ and $s_3 \le (r-1)/r$.

If $rs_1 \ge s_2$, $rs_1 + (r-1)s_2 \ge s_3$ and $s_1 < \frac{1}{r+1}$, we can guarantee that $b_2$, $b_3$ and $p_1$ will not occur.

If $s_3 \le \frac{r-1}{r} \Leftrightarrow \frac{s_2+s_3+s_1}{v_1} = \frac{1}{v_1} \le \frac{1-s_3}{v_2} = \frac{s_1+s_2}{v_2}$, worker 1 can complete an item and reach the start of station 2 before worker 2 reaches the end of station 2. Thus, $p_2$ will occur.

(e) $a^{(t+1)} = p_3$. In this subcase, we can guarantee that $b_2$, $b_3$, $p_1$ and $p_2$ will not occur.

Due to $r \ge 2 \Leftrightarrow s_2 < 1 \le r-1 \Leftrightarrow \frac{s_2+s_3+s_1+s_2}{v_1} = \frac{1+s_2}{v_1} < \frac{1}{v_2} = \frac{s_1+s_2+s_3}{v_2}$, worker 1 can complete an item and reach the start of station 3 before worker 2 reaches the end of station 3. Thus, $p_3$ will occur.

Following the same method as described for case (A), the function $f$ for case (B) and case (C) can be constructed. □

✧ **Proof of LEMMA 4.**

We define the following twelve lines $l_1 \sim l_{12}$, as illustrated in Figure A.1. Each line represents a combination of work content distribution and velocity ratio for function $a^{(t+1)} = f(a^{(t)})$ when $r \ge 2$. Let

$l_1$: $s_2 = rs_1$; $l_2$: $s_2 = \frac{1}{r+1} - \frac{1}{r+1}s_1 \Leftrightarrow rs_2 = s_3$; $l_3$: $s_2 = 1 - \frac{r+1}{r}s_1 \Leftrightarrow rs_3 = s_1$;

$l_4$: $s_2 = \frac{1}{r} - \frac{r+1}{r}s_1 \Leftrightarrow rs_1 + (r-1)s_2 = s_3$;

$l_5$: $s_2 = 1 - r + rs_1 \Leftrightarrow rs_2 + (r-1)s_3 = s_1$;

$l_6$: $s_2 = \frac{r}{r+1} - \frac{1}{r+1}s_1 \Leftrightarrow rs_3 + (r-1)s_1 = s_2$;

$l_7$: $s_1 = \frac{r-1}{r}$; $l_8$: $s_2 = \frac{r-1}{r}$; $l_9$: $s_1 + s_2 = \frac{1}{r} \Leftrightarrow s_3 = \frac{r-1}{r}$;

$l_{10}$: $s_1 = \frac{1}{r+1}$; $l_{11}$: $s_2 = \frac{1}{r+1}$; $l_{12}$: $s_2 = \frac{r}{r+1} - s_1 \Leftrightarrow s_3 = \frac{1}{r+1}$.

We divide the entire feasible work content area into seven regions, as displayed in Figure 4(a). If $r \ge 2$, lines $l_1 \sim l_9$ intersect with region 1~6, as demonstrated in Figure A.1, which represent different work content distributions. We check the function $a^{(t+1)} = f(a^{(t)})$ to determine cycles and throughputs of a rotating *seru* in each region separately. Let

$\theta_1 = p_1c_1p_1$, $\theta_2 = p_2c_1p_2$, $\theta_3 = p_3c_1p_3$. And $\tau_1 = v_2/s_1$, $\tau_2 = v_2/s_2$, $\tau_3 = v_2/s_3$.

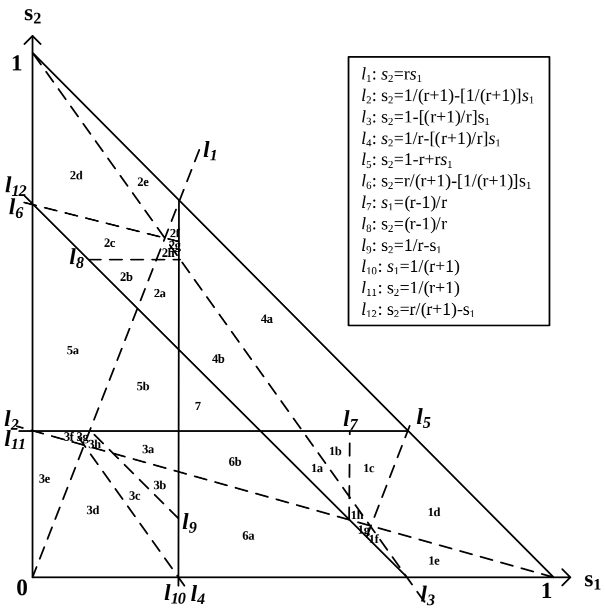



Figure A.1. Seven regions intersect with lines $l_1 \sim l_9$ when $r \geq 2$.

**Region 1,** This region is partitioned into eight mutually exclusive sub regions 1a~1h by lines $l_2$, $l_3$, $l_5$ and $l_7$. We construct cycles by tracking all feasible initial iterates $a^{(0)}$ in each sub region.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For all the eight sub regions, we have $f(a^{(0)}) = p_1$ because $s_1 \geq \frac{1}{r+1}$. Thus, the orbit $p_1 \to f(p_1): p_1 \to \cdots$ or $b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(B) If $a^{(0)} \in \{p_3, b_3\}$,

(a) For sub regions 1b, 1c, 1d and 1e,

We have $f(a^{(0)}) = b_1$ because $s_2 > 1 - \frac{r+1}{r} s_1$. Combined with case (A), the orbit $p_3 \to f(p_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(b) For sub regions 1a, 1f, 1g and 1h,

We have $f(a^{(0)}) = p_1$ because $s_2 \leq 1 - \frac{r+1}{r} s_1$, $s_2 \leq \frac{r}{r+1} - \frac{1}{r+1} s_1$, $s_1 + s_2 > \frac{r}{r+1}$, and $s_2 \leq \frac{r-1}{r}$. Combined with case (A), the orbit $p_3 \to f(p_3): p_1 \to f(p_1): p_1 \to \cdots$ or $b_3 \to f(b_3): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(C) If $a^{(0)} \in \{p_2, b_2\}$,

(a) For sub regions 1e, 1f and 1g,

We have $f(a^{(0)}) = b_3$ because $s_2 < \frac{1}{r+1} - \frac{1}{r+1}s_1$.

(I) For sub region 1e,

Combined with case (A) and case (B)(a), the orbit $p_2 \to f(p_2): b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(II) For sub regions 1f and 1g,

Combined with case (A) and case (B)(b), the orbit $p_2 \to f(p_2): b_3 \to f(b_3): p_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_3 \to f(b_3): p_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(b) For sub region 1d,

We have $f(a^{(0)}) = b_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1$ and $s_2 < 1 - r + rs_1$.

Combined with case (A), the orbit $p_2 \to f(p_2): b_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(c) For sub regions 1a and 1b,

We have $f(a^{(0)}) = p_3$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1$, $s_2 \geq 1 - r + rs_1$, $s_2 < \frac{1}{r+1}$ and $s_1 \leq \frac{r-1}{r}$.

(I) For sub region 1b,

Combined with case (A) and case (B)(a), the orbit $p_2 \to f(p_2): p_3 \to f(p_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): p_3 \to f(p_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(II) For sub region 1a,

Combined with case (A) and case (B)(b), the orbit $p_2 \to f(p_2): p_3 \to f(p_3): p_1 \to f(p_1): p_1 \to \cdots$ or $b_2 \to f(b_2): p_3 \to f(p_3): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(d) For sub regions 1c and 1h,

We have $f(a^{(0)}) = p_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1, s_2 \geq 1 - r + rs_1, s_2 < \frac{1}{r+1}$ and $s_1 > \frac{r-1}{r}$. Combined with case (A), the orbit $p_2 \to f(p_2): p_1 \to f(p_1): p_1 \to \cdots$ or $b_2 \to f(b_2): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

Therefore, given any initial iterate $a^{(0)}$, the cycle in Region 1 is $\theta_1$. Similarly, we conclude that given any $a^{(0)}$, the cycles in Regions 2 and 3 are $\theta_2$ and $\theta_3$, respectively.

**Region 4**, This region is partitioned into two mutually exclusive sub regions 4a and 4b, by line $l_3$. We construct cycles by tracking all feasible initial iterates $a^{(0)}$ in each sub region.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For both sub regions 4a and 4b, we have $f(a^{(0)}) = p_1$ because $s_1 \geq \frac{1}{r+1}$. Thus, the orbit $p_1 \to f(p_1){:}\, p_1 \to \cdots$ or $b_1 \to f(b_1){:}\, p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(B) If $a^{(0)} \in \{p_2, b_2\}$,

For both sub regions 4a and 4b, we have $f\left(a^{(0)}\right) = p_2$ because $s_2 \geq \frac{1}{r+1}$. Thus, the orbit $p_2 \to f(p_2){:}\, p_2 \ldots$ or $b_2 \to f(b_2){:}\, p_2 \ldots$ is obtained.

——In this case, $\theta_2$ is constructed with throughput $\tau_2$.

(C) If $a^{(0)} \in \{p_3, b_3\}$,

(a) For sub region 4a,

We have $f(a^{(0)}) = b_1$ because $s_2 > 1 - \frac{r+1}{r}s_1$. Combined with case (A), the orbit $p_3 \to f(p_3){:}\, b_1 \to f(b_1){:}\, p_1 \to \cdots$ or $b_3 \to f(b_3){:}\, b_1 \to f(b_1){:}\, p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(b) For sub region 4b,

We have $f(a^{(0)}) = p_1$ because $s_2 \leq 1 - \frac{r+1}{r}s_1$, $s_2 \leq \frac{r}{r+1} - \frac{1}{r+1}s_1$, $s_1 + s_2 > \frac{r}{r+1}$ and $s_2 \leq \frac{r-1}{r}$. Combined with case (A), the orbit $p_3 \to f(p_3){:}\, p_1 \to f(p_1){:}\, p_1 \to \cdots$ or $b_3 \to f(b_3){:}\, p_1 \to f(p_1){:}\, p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

Therefore, in Region 4, if $a^{(0)} \in \{p_1, p_3, b_1, b_3\}$, the cycle is $\theta_1$; if $a^{(0)} \in \{p_2, b_2\}$, the cycle is $\theta_2$. Similarly, in Region 5, if $a^{(0)} \in \{p_1, p_2, b_1, b_2\}$, the cycle is $\theta_2$; if $a^{(0)} \in \{p_3, b_3\}$, the cycle is $\theta_3$. In Region 6, if $a^{(0)} \in \{p_2, p_3, b_2, b_3\}$, the cycle is $\theta_3$; if $a^{(0)} \in \{p_1, b_1\}$, the cycle is $\theta_1$.

**Region 7**, This region will never intersect with lines $l_1 \sim l_9$.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For region 7, we have $f(a^{(0)}) = p_1$ because $s_1 \geq \frac{1}{r+1}$. As a result, the orbit $p_1 \to f(p_1){:}\, p_1 \to \cdots$ or $b_1 \to f(b_1){:}\, p_1 \to \cdots$ is obtained.

——In this case, $\theta_1$ is constructed with throughput $\tau_1$.

(B) If $a^{(0)} \in \{p_2, b_2\}$,

For region 7, we have $f(a^{(0)}) = p_2$ because $s_2 \geq \frac{1}{r+1}$. Thus, the orbit $p_2 \rightarrow f(p_2): p_2 \rightarrow \cdots$ or $b_2 \rightarrow f(b_2): p_2 \rightarrow \cdots$ is obtained.

——In this case, $\theta_2$ is constructed with throughput $\tau_2$.

(C) If $a^{(0)} \in \{p_3, b_3\}$,

For region 7, we have $f(a^{(0)}) = p_3$ because $s_1 + s_2 \leq \frac{r}{r+1}$. Thus, the orbit $p_3 \rightarrow f(p_3): p_3 \rightarrow \cdots$ or $b_3 \rightarrow f(b_3): p_3 \rightarrow \cdots$ is obtained.

——In this case, $\theta_3$ is constructed with throughput $\tau_3$.

Therefore, for Region 7, if $a^{(0)} \in \{p_1, b_1\}$, the cycle is $\theta_1$; if $a^{(0)} \in \{p_2, b_2\}$, the cycle is $\theta_2$; if $a^{(0)} \in \{p_3, b_3\}$, the cycle is $\theta_3$.

In the cycles $\theta_1$, $\theta_2$ and $\theta_3$, $c_2$ never occurs. Worker 2 consistently works on a unique station. Correspondingly, for $\theta_1$, the throughput is $\tau_1 = v_2/s_1$; for $\theta_2$, the throughput is $\tau_2 = v_2/s_2$; for $\theta_3$, the throughput is $\tau_1 = v_2/s_3$. □

✧ **Proof of THEOREM 2.**

There are three exclusive cases.

Case (1): $a^{(0)} = b_1 \, or \, p_1$.

(A) In Figure 4, for region 1, 4, 6, and 7, the cycle is $\theta_1$, and its throughput is $\tau_1$. Given the velocity of worker 2, $v_2$, the throughput $\tau_1$ increases as the work amount on station 1, $s_1$, decreases. When $s_1 = \frac{1}{r+1}$, the maximal throughput is achieved, which is $\frac{v_2}{s_1} = \frac{v_2}{1/(r+1)} = v_1 + v_2$.

(B) For regions 2 and 5, the cycle is $\theta_2$, and its throughput is $\tau_2$. Given the velocity of worker 2, $v_2$, the throughput $\tau_2$ increases as the work amount on station 2, $s_2$, decreases. When $\mathrm{s}_2 = \frac{1}{r+1}$, the maximal throughput is achieved, which is $\frac{v_2}{s_2} = \frac{v_2}{1/(r+1)} = v_1 + v_2$.

(C) For region 3, the cycle is $\theta_3$, and its throughput is $\tau_3$. Given the velocity of worker 2, $v_2$, the throughput $\tau_3$ increases the work amount on station 3, $s_3$, decreases. When $s_3 = \frac{r-1}{r+1} \Leftrightarrow 1 - s_3 = s_1 + s_2 = \frac{2}{r+1}$, the maximal throughput is achieved, which is $\frac{v_2}{s_3} = \frac{v_2}{(r-1)/(r+1)} < v_1 + v_2$ because $r \geq 2$.

Therefore, if the initial iterates $a^{(0)} = b_1 \, or \, p_1$, the maximal throughput is achieved when $s_1 = \frac{1}{r+1}$; or $\mathrm{s}_2 = \frac{1}{r+1}$ and $0 < \mathrm{s}_1 < \frac{1}{r+1}$. In this case, the maximal throughput is $v_1 + v_2$.

Using the same method as described for Case (1), the maximal throughput for

Case (2) and Case (3) can be obtained.

For Case (2), the maximal throughput is achieved when $s_2 = \frac{1}{r+1}$; or $s_3 = \frac{1}{r+1} \Leftrightarrow s_1 + s_2 = \frac{r}{r+1}$ and $0 < s_2 < \frac{1}{r+1}$. In this case, the maximal throughput is $v_1 + v_2$.

For Case (3), the maximal throughput is achieved when $s_3 = \frac{1}{r+1} \Leftrightarrow s_1 + s_2 = \frac{r}{r+1}$; or $s_1 = \frac{1}{r+1}$ and $0 < s_3 < \frac{1}{r+1}$. In this case, the maximal throughput is $v_1 + v_2$. □

✧ **<u>Proof of LEMMA 5.</u>**

We construct the function $f$ separately for the following three cases: (A) $a^{(t)} \in \{p_1, b_1\}$, (B) $a^{(t)} \in \{p_2, b_2\}$, and (C) $a^{(t)} \in \{p_3, b_3\}$. We determine the next iterate $a^{(t+1)}$ by considering all possible combinations of work content distributions on stations.

(A) $a^{(t)} \in \{p_1, b_1\}$

The positions where worker 1 and worker 2 are located immediately after $p_1$ or $b_1$ are the end and the start of station 1, respectively. Since $s_1 < \frac{k-(k-1)r}{r+1} \leq \frac{1}{r+1} \Rightarrow \frac{s_1}{v_2} < \frac{1-s_1}{v_1}$, worker 2 departs from the end of station 1 before worker 1 can complete an item and return to the start of station 1.

(a) $a^{(t+1)} = b_2$ if $rs_1 < s_2$.

If $rs_1 < s_2 \Leftrightarrow \frac{s_2}{v_1} > \frac{s_1}{v_2}$, worker 2 can reach the start of station 2 before worker 1 finishes his work on station 2. Thus, $b_2$ will occur.

(b) $a^{(t+1)} = b_3$ if $rs_1 \geq s_2, rs_1 + (r-1)s_2 < s_3$.

If $rs_1 \geq s_2$, $b_2$ will not occur.

If $rs_1 + (r-1)s_2 < s_3 \Leftrightarrow \frac{s_1+s_2}{v_2} < \frac{s_2+s_3}{v_1}$, worker 2 can reach the start of station 3 before worker 1 finishes his work on station 3. Thus, $b_3$ will occur.

(c) $a^{(t+1)} = p_2$ if $rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3, s_3 \leq \Omega(k)$.

If $rs_1 \geq s_2$ and $rs_1 + (r-1)s_2 \geq s_3$, $b_2$ and $b_3$ will not occur.

Since $s_1 < \Phi(k), s_2 < \Phi(k)$ and $s_3 < \Omega(k)$, $p_1, p_2$ and $p_3$ will not occur before worker 2 completes $k-1$ items and reaches the end of station 1.

Assume the opposite; let $j$ be the index of that station where passing occurs before worker 2 completes $k-1$ items and then reaches the end of station 1.

There are three exclusive cases: (1) $j = 1$, (2) $j = 2$ and (3) $j = 3$.

(1) $j = 1$

The time for worker 1 to complete $k'(k' \leq k)$ items and then reach the start of station 1 is $t_1 = \frac{s_2+s_3+k'-1}{v_1}$. The time for worker 2 to complete $k'-1$ items and then reach the end of station 1 is $t_2 = \frac{s_1+s_2+s_3+k'-2+s_1}{v_2}$. Since we assume that passing occurs on station 1, we have $t_1 \le t_2 \Longrightarrow s_1 \ge \frac{k'-(k'-1)r}{r+1} \ge \frac{k-(k-1)r}{r+1} = \Phi(k)$, which contradicts $s_1 < \Phi(k)$.

(2) $j = 2$

The time for worker 1 to complete $k'(k' \le k-1)$ items and then reach the start of station 2 is $t_1 = \frac{s_2+s_3+k'-1+s_1}{v_1}$. The time for worker 2 to complete $k'-1$ items and then reach the end of station 2 is $t_2 = \frac{s_1+s_2+s_3+k'-2+s_1+s_2}{v_2}$. Since we assume that passing occurs on station 2, we have $t_1 \le t_2 \Longrightarrow s_3 \le \frac{r-1}{r}k' \le \frac{r-1}{r}(k-1)$, which contradicts $s_3 = 1 - s_1 - s_2 > 1 - 2\Phi(k) > \frac{r-1}{r}(k-1)$.

(3) $j = 3$

The time for worker 1 to complete $k'(k' \le k-1)$ items and then reach the start of station 3 is $t_1 = \frac{s_2+s_3+k'-1+s_1+s_2}{v_1}$. The time for worker 2 to complete $k'-1$ items and then reach the end of station 3 is $t_2 = \frac{s_1+s_2+s_3+k'-2+s_1+s_2+s_3}{v_2}$. Since we assume that passing occurs on station 3, we have $t_1 \le t_2 \Longrightarrow s_2 \le k'(r-1) \le (r-1)(k-1)$, which contradicts $s_2 = 1 - s_1 - s_3 > 1 - \Phi(k) - \Omega(k) > (r-1)(k-1)$.

If $s_3 \le \Omega(k) \Leftrightarrow \frac{s_2+s_3+k-1+s_1}{v_1} = \frac{k}{v_1} \le \frac{k-s_3}{v_2} = \frac{s_1+s_2+s_3+k-2+s_1+s_2}{v_2}$, worker 1 can complete $k$ items and reach the start of station 2 before worker 2 completes $k-1$ items and then reaches the end of station 2. Thus, $p_2$ will occur.

(d) $a^{(t+1)} = p_3$ if $rs_1 \ge s_2, rs_1 + (r-1)s_2 \ge s_3, s_3 > \Omega(k), s_2 \le r\Omega(k)$.

If $rs_1 \ge s_2$, $rs_1 + (r-1)s_2 \ge s_3$ and $s_3 > \Omega(k)$, $b_2, b_3, p_1, p_2$ and $p_3$ will not occur before worker 2 completes $k-1$ items and then reaches the end of station 2.

If $s_2 \le r\Omega(k) \Leftrightarrow \frac{s_2+s_3+k-1+s_1+s_2}{v_1} = \frac{k+s_2}{v_1} \le \frac{k}{v_2} = \frac{s_1+s_2+s_3+k-2+s_1+s_2+s_3}{v_1}$, worker 1 can complete $k$ items and then reach the start of station 3 before worker 2 completes $k-1$ items and then reaches the end of station 3. Thus, $p_3$ will occur.

(e) $a^{(t+1)} = p_1$ if $rs_1 \ge s_2$, $rs_1 + (r-1)s_2 \ge s_3$, $s_3 > \Omega(k)$, $s_2 > r\Omega(k)$, $s_1 \ge$

$\Phi(k+1)$.

If $rs_1 > s_2, rs_1 + (r-1)s_2 > s_3, s_3 > \Omega(k), s_2 > r\Omega(k)$, $b_2, b_3, p_1, p_2$ and $p_3$ will not occur before worker 2 completes $k-1$ items and then reaches the end of station 3.

If $s_1 \geq \Phi(k+1) \Leftrightarrow \frac{s_2+s_3+k}{v_1} = \frac{k+1-s_1}{v_1} \leq \frac{k+s_1}{v_2} = \frac{s_1+s_2+s_3+k-1+s_1}{v_2}$, worker 1 can complete $k+1$ items and then reach the start of station 1 before worker 2 completes $k$ items and reaches the end of station 1. Thus, $p_1$ will occur.

(f) $a^{(t+1)} = p_2$ if $rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3$, $s_3 > \Omega(k), s_2 > r\Omega(k)$, $s_1 < \Phi(k+1), s_2 \geq \Phi(k+1)$.

If $rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3$, $s_3 \geq \Omega(k), s_2 > r\Omega(k)$ and $s_1 < \Phi(k+1)$, $b_2, b_3, p_1, p_2$ and $p_3$ will not occur before worker 2 completes $k$ items and then reaches the end of station 1.

If $s_1 < \Phi(k+1)$ and $s_2 < rs_1$, we have $s_3 = 1 - s_1 - s_2 > (r-1)k > \frac{r-1}{r}(k+1) \Longrightarrow \frac{s_2+s_3+k+s_1}{v_1} < \frac{s_1+s_2+s_3+k-1+s_1+s_2}{v_2}$. Worker 1 can complete *k+1* items and then reach the start of station 2 before worker 2 completes $k$ items and reaches the end of station 2. Thus, $p_2$ will occur.

(g) $a^{(t+1)} = p_3$ if $rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3, s_3 > \Omega(k), s_2 > r\Omega(k)$, $s_1 < \Phi(k+1), s_2 < \Phi(k+1), s_3 \geq \Phi(k+1)$.

If $rs_1 \geq s_2, rs_1 + (r-1)s_2 \geq s_3, s_3 > \Omega(k), s_2 > r\Omega(k)$, $s_1 < \Phi(k+1), s_2 < \Phi(k+1)$, $b_2, b_3, p_1, p_2$ and $p_3$ will not occur before worker 2 completes $k$ items and then reaches the end of station 2.

If $s_3 \geq \Phi(k+1)$ and $rs_1 \geq s_2$, we have $s_2 \leq (k+1)(r-1) \Longrightarrow \frac{s_2+s_3+k+s_1+s_2}{v_1} \leq \frac{s_1+s_2+s_3+k-1+s_1+s_2+s_3}{v_2}$. Worker 1 can complete $k+1$ items and then reach the start of station 3 before worker 2 completes k items and reaches the end of station 3. Thus, $p_3$ will occur.

Using the same method as described for case (A), the function $f$ for case (B) and case (C) can be constructed. □

✧ **<u>Proof of LEMMA 6.</u>**

We define the following twelve lines $l_1 \sim l_{15}$, as shown in Figure A.2. Each line represents some iterate conditions of the function $a^{(t+1)} = f(a^{(t)})$ for $D(k)$, $k = k^* - 1$. Let

$l_1$: $s_2 = rs_1$; $l_2$: $s_2 = \frac{1}{r+1} - \frac{1}{r+1}s_1 \Leftrightarrow rs_2 = s_3$; $l_3$: $s_2 = 1 - \frac{r+1}{r}s_1 \Leftrightarrow rs_3 = s_1$;

$l_4$: $s_2 = \frac{1}{r} - \frac{r+1}{r}s_1 \Leftrightarrow rs_1 + (r-1)s_2 = s_3$;

$l_5$: $s_2 = 1 - r + rs_1 \Leftrightarrow rs_2 + (r-1)s_3 = s_1$;

$l_6$: $s_2 = \frac{r}{r+1} - \frac{1}{r+1} s_1 \Leftrightarrow rs_3 + (r-1)s_1 = s_2$;

$l_7$: $s_1 = \frac{k+1-kr}{r+1} \Leftrightarrow s_1 = \Phi(k+1)$; $l_8$: $s_2 = \frac{k+1-kr}{r+1} \Leftrightarrow s_2 = \Phi(k+1)$;

$l_9$: $s_2 = \frac{(k+1)r-k}{r+1} - s_1 \Leftrightarrow s_3 = \Phi(k+1)$;

$l_{10}$: $s_2 = \frac{k-(k-1)r}{r} - s_1 \Leftrightarrow s_3 = \Omega(k)$; $l_{11}$: $s_2 = k+1-kr-s_1 \Leftrightarrow s_3 = r\Omega(k)$;

$l_{12}$: $s_1 = \frac{k(r-1)}{r} \Leftrightarrow s_1 = \Omega(k)$; $l_{13}$: $s_1 = k(r-1) \Leftrightarrow s_1 = r\Omega(k)$;

$l_{14}$: $s_2 = \frac{k(r-1)}{r} \Leftrightarrow s_3 = \Omega(k)$; $l_{15}$: $s_2 = k(r-1) \Leftrightarrow s_2 = r\Omega(k)$.

We partition the entire feasible work content area into twelve mutually exclusive regions as Figure 8. $l_1 \sim l_{15}$ intersect with $D(k), k = k^* - 1$ (see Figure A.2), which means distinct stations' distributions. We study the function $a^{(t+1)} = f(a^{(t)})$ to determine the cycles and throughputs of the system in each region separately. Let

$\theta_1(k) = p_1(c_1c_2)^{k-1}c_1p_2(c_1c_2)^k c_1p_1$, $\tau_1(k) = \frac{4k}{2k-1+s_1+s_2} \times v_2$,

$\theta_2(k) = p_2(c_1c_2)^{k-1}c_1p_3(c_1c_2)^k c_1p_2$, $\tau_2(k) = \frac{4k}{2k-1+s_2+s_3} \times v_2$,

$\theta_3(k) = p_1(c_1c_2)^{k-1}c_1p_3(c_1c_2)^k c_1p_1$, $\tau_3(k) = \frac{4k}{2k-1+s_1+s_3} \times v_2$,

$\theta_4(k) = p_1b_2(c_1c_2)^k c_1p_1$, $\tau_4(k) = \frac{2k+1}{k+s_2/r} \times v_2$,

$\theta_5(k) = p_2b_3(c_1c_2)^k c_1p_2$, $\tau_5(k) = \frac{2k+1}{k+s_3/r} \times v_2$,

$\theta_6(k) = p_3c_1c_2b_1(c_1c_2)^{k-1}c_1p_3$, $\tau_6(k) = \frac{2k+1}{k+s_1/r} \times v_2$,

$\theta_7(k) = p_1(c_1c_2)^k c_1p_1$, $\tau_7(k) = \frac{2k+1}{k+s_1} \times v_2$,

$\theta_8(k) = p_2(c_1c_2)^k c_1p_2$, $\tau_8(k) = \frac{2k+1}{k+s_2} \times v_2$,

$\theta_9(k) = p_3(c_1c_2)^k c_1p_3$, $\tau_9(k) = \frac{2k+1}{k+s_3} \times v_2$.

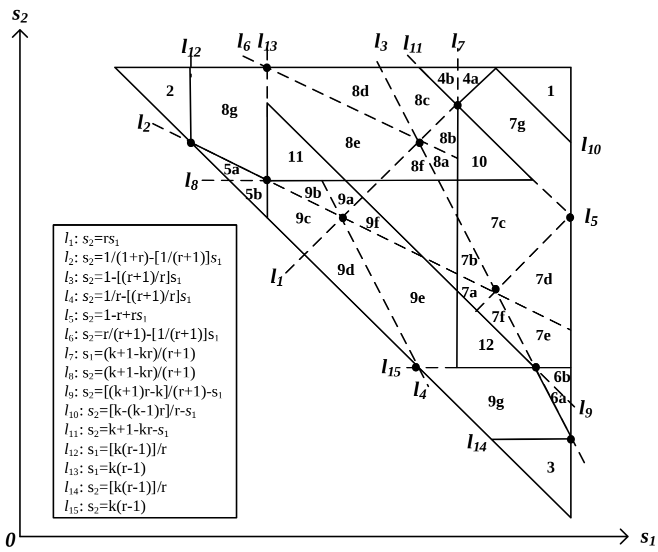



Figure A.2 Twelve regions intersected with lines $l_1 \sim l_{15}$ for $D(k),\ k = k^* - 1$.

**Region k1**, The region will never interact with line $l_1 \sim l_6$.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For region k1, we have $f(a^{(0)}) = p_2$ because $rs_1 \geq s_2, s_2 \geq \frac{1}{r} - \frac{r+1}{r} s_1$ and $s_1 + s_2 \geq \frac{k-(k-1)r}{r}$.

(B) If $a^{(0)} \in \{p_2, b_2\}$,

For region k1, we have $f(a^{(0)}) = p_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1} s_1, s_2 \geq 1 - r + rs_1, s_1 >$

$\frac{k(r-1)}{r}$ and $s_1 + s_2 \leq k + 1 - kr$.

(C) If $a^{(0)} \in \{p_3, b_3\}$,

For region k1, we have $f(a^{(0)}) = b_1$ because $s_2 > 1 - \frac{r+1}{r} s_1$.

Combined with case (A), case (B), and case (C), we have the following feasible orbits: If $a^{(0)} \in \{p_1, b_1\}$, the orbit $p_1 \to f(p_1){:}\, p_2 \to f(p_2){:}\, p_1 \to \cdots$ or $b_1 \to f(b_1){:}\, p_2 \to f(p_2){:}\, p_1 \to \cdots$ is obtained; If $a^{(0)} \in \{p_2, b_2\}$, the orbit $p_2 \to f(p_2){:}\, p_1 \to f(p_1){:}\, p_2 \to \cdots$ or $b_2 \to f(b_2){:}\, p_1 \to f(p_1){:}\, p_2 \to \cdots$ is obtained; If $a^{(0)} \in \{p_3, b_3\}$, the orbit $p_3 \to f(p_3){:}\, b_1 \to f(b_1){:}\, p_2 \to f(p_2){:}\, p_1 \to \cdots$ or $b_3 \to f(b_3){:}\, b_1 \to f(b_1){:}\, p_2 \to f(p_2){:}\, p_1 \to \cdots$ is obtained. In all the above three cases, $\theta_1(k)$ is constructed. Similarly, the cycles in Region k2 and Region k3 are $\theta_2(k)$ and $\theta_3(k)$, respectively.

**Region k4**, The region will never interact with line $l_1 \sim l_6$.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For region k4, we have $f(a^{(0)}) = b_2$ because $s_2 > rs_1$.

(B) If $a^{(0)} \in \{p_2, b_2\}$,

For region k4, we have $f\left(a^{(0)}\right) = p_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1, s_2 \geq 1 - r + rs_1, s_1 > \frac{k(r-1)}{r}$ and $s_1 + s_2 \geq k + 1 - kr$.

(C) If $a^{(0)} \in \{p_3, b_3\}$,

For region k4, we have $f\left(a^{(0)}\right) = b_1$ because $s_2 > 1 - \frac{r+1}{r}s_1$.

Combined with case (A), case (B), and case (C), we have the following feasible orbits. If $a^{(0)} \in \{p_1, b_1\}$, the orbit $p_1 \to f(p_1): b_2 \to f(b_2): p_1 \to \cdots$ or $b_1 \to f(b_1): b_2 \to f(b_2): p_1 \to \cdots$ is obtained; If $a^{(0)} \in \{p_2, b_2\}$, the orbit $p_2 \to f(p_2): p_1 \to f(p_1): b_2 \to \cdots$ or $b_2 \to f(b_2): p_1 \to f(p_1): b_2 \to \cdots$ is obtained; If $a^{(0)} \in \{p_3, b_3\}$, the orbit $p_3 \to f(p_3): b_1 \to f(b_1): b_2 \to f(b_2): p_1 \to \cdots$ or $b_3 \to f(b_3): b_1 \to f(b_1): b_2 \to f(b_2): p_1 \to \cdots$ is obtained. In the above three cases, $\theta_4(k)$ is constructed. Similarly, the cycles in Region k5 and Region k6 are $\theta_5(k)$ and $\theta_6(k)$, respectively.

**Region k7**, This region is partitioned into seven sub regions 7a~7g by lines $l_2, l_3, l_5$ and $l_8$. We construct cycles by tracking all feasible initial behaviors in each sub region.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

For all seven sub regions, we have $f\left(a^{(0)}\right) = p_1$ because $rs_1 \geq s_2, s_2 \geq \frac{1}{r} - \frac{r+1}{r}s_1, s_1 + s_2 < \frac{k-(k-1)r}{r}, s_2 > k(r-1)$ and $s_1 \geq \frac{k+1-kr}{r+1}$. Thus, the orbit $p_1 \to f(p_1): p_1 \to \cdots$ or $b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed in region k7.

(B) If $a^{(0)} \in \{p_3, b_3\}$,

(a) For sub regions 7c, 7d, 7e, 7g,

We have $f\left(a^{(0)}\right) = b_1$ because $s_2 > 1 - \frac{r+1}{r}s_1$. Combined with case (A), the orbit $p_3 \to f(p_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(b) For sub regions 7a, 7b and 7f,

We have $f\left(a^{(0)}\right) = p_1$ because $s_2 \leq 1 - \frac{r+1}{r}s_1, s_2 \geq \frac{r}{r+1} - \frac{1}{r+1}s_1, s_2 > \frac{k(r-1)}{r}, s_1 > k(r-1), s_1 + s_2 > \frac{(k+1)r-k}{r+1}$ and $s_1 \geq \frac{k+1-kr}{r+1}$. Combined with case (A), the orbit $p_3 \to f(p_3): p_1 \to f(p_1): p_1 \to \cdots$ or $b_3 \to f(b_3): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(C) If $a^{(0)} \in \{p_2, b_2\}$,

(a) For sub regions 7a, 7f, 7e,

We have $f(a^{(0)}) = b_3$ because $s_2 < \frac{1}{r+1} - \frac{1}{r+1}s_1$.

(I) For sub region 7e,

Combined with case (A) and case (B)(a), the orbit $p_2 \to f(p_2): b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(II) For sub regions 7a and 7f

Combined with case (A) and case (B)(b), the orbit $p_2 \to f(p_2): b_3 \to f(b_3): p_1 \to f(p_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_3 \to f(b_3): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(b) For sub region 7d,

We have $f(a^{(0)}) = b_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1$ and $s_2 < 1 - r + rs_1$.

Combined with case (A), the orbit $p_2 \to f(p_2): b_1 \to f(b_1): p_1 \to \cdots$ or $b_2 \to f(b_2): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(c) For sub region 7g,

We have $f(a^{(0)}) = p_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1, s_2 \geq 1 - r + rs_1, s_1 > \frac{k(r-1)}{r}$ and $s_1 + s_2 \geq k + 1 - kr$. Combined with case (A), the orbit $p_2 \to f(p_2): p_1 \to f(p_1): p_1 \to \cdots$ or $b_2 \to f(b_2): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(e) For sub regions 7b and 7c,

We have $f(a^{(0)}) = p_1$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1, s_2 \geq 1 - r + rs_1, s_1 > \frac{k(r-1)}{r}, s_1 + s_2 < k + 1 - kr, s_2 < \frac{k+1-kr}{r+1}$ and $s_3 \leq \frac{k+1-kr}{r+1}$. Combined with case (A), the orbit $p_2 \to f(p_2): p_1 \to f(p_1): p_1 \to \cdots$ or $b_2 \to f(b_2): p_1 \to f(p_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

Therefore, for Region k7, given any initial iterate, the cycle is $\theta_7(k)$. Similarly, we conclude that given any initial iterate, the cycles in Region k8 and Region k9 are $\theta_8(k)$ and $\theta_9(k)$, respectively.

**Region k10**, The region will never interact with line $l_1 \sim l_6$.

(A) If $a^{(0)} \in \{p_1, b_1\}$,

We have $f(a^{(0)}) = p_1$ because $rs_1 \geq s_2, s_2 \geq \frac{1}{r} - \frac{r+1}{r}s_1, s_1 + s_2 < \frac{k-(k-1)r}{r}, s_2 > k(r-1)$ and $s_1 \geq \frac{k+1-kr}{r+1}$. Thus, the orbit $p_1 \to f(p_1): p_1 \to \cdots$ or $b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

(B) If $a^{(0)} \in \{p_2, b_2\}$,

We have $f(a^{(0)}) = p_2$ because $s_2 \geq \frac{1}{r+1} - \frac{1}{r+1}s_1, s_2 \geq 1 - r + rs_1, s_1 > \frac{k(r-1)}{r}, s_1 + s_2 < k + 1 - kr$ and $s_2 \geq \frac{k+1-kr}{r+1}$. Thus, the orbit $p_2 \to f(p_2): p_2 \to \cdots$ or $b_2 \to f(b_2): p_2 \to \cdots$ is obtained.

——In this case, $\theta_8(k)$ is constructed.

(C) If $a^{(0)} \in \{p_3, b_3\}$,

We have $f(a^{(0)}) = b_1$ because $s_2 > 1 - \frac{r+1}{r}s_1$. Combined with case (A), the orbit $p_3 \to f(p_3): b_1 \to f(b_1): p_1 \to \cdots$ or $b_3 \to f(b_3): b_1 \to f(b_1): p_1 \to \cdots$ is obtained.

——In this case, $\theta_7(k)$ is constructed.

Therefore, if $a^{(0)} \in \{p_1, p_3, b_1, b_3\}$, the cycle in the region k10 is $\theta_7(k)$; if $a^{(0)} \in \{p_2, b_2\}$, the cycle in the region is $\theta_8(k)$. Similarly, if $a^{(0)} \in \{p_1, p_2, b_1, b_2\}$, the cycle in the region k11 is $\theta_8(k)$; if $a^{(0)} \in \{p_3, b_3\}$, the cycle in the region is $\theta_9(k)$. If $a^{(0)} \in \{p_2, p_3, b_2, b_3\}$, the cycle in the region k12 is $\theta_9(k)$; if $a^{(0)} \in \{p_1, b_1\}$, the cycle in the region is $\theta_7(k)$.

In cycle $\theta_1(k)$, worker 1 assembles $2k+1$ items, and worker 2 assembles $2k-1$ items. The total number of items completed by the two workers is $4k$, which takes time $(2k - 1 + s_1 + s_2)/v_2$, as worker 2 remains busy. Therefore, the throughput of the rotating *seru* is $\tau_1(k) = \frac{4k}{2k-1+s_1+s_2} \times v_2$. Similarly, we can obtain the corresponding throughputs of other cycles $\theta_1(k)$~ $\theta_9(k)$. □

✧ **Details of LEMMA 7.**

There are 15 sub regions formed based on the decreasing velocity ratios, as illustrated in Figure A.3. For example, if $\Omega(k^*) \geq \Phi(k^*)$, only region k*0 exists; if $\Phi(k^*) > \Omega(k^*) > 1/3$, four regions are present: k*0, k*1, k*2 and k*3. Region k*0, k*0′, and k*1~k*9 correspond to one cycle, while Region k*10~k*12 represent two feasible cycles; Region k*13 is associated with three feasible cycles. In addition to $\theta_1(k^*)$~$\theta_9(k^*)$, two new feasible cycles appear in $D(k^*)$, which are:

$$\theta_0(k^*) = p_1(c_1c_2)^{k^*-1}c_1p_2(c_1c_2)^{k^*-1}c_1p_3(c_1c_2)^{k^*}c_1p_1,\ \tau_0(k^*) = \frac{6k^*-1}{3k^*-1} \times v_2;$$

$$\theta_0'(k^*) = p_1(c_1c_2)^{k^*-1}c_1p_3(c_1c_2)^{k^*}c_1p_2(c_1c_2)^{k^*}c_1p_1,\ \tau_0'(k^*) = \frac{6k^*+1}{3k^*} \times v_2.$$

Region k*0: The cycle is $\theta_0(k^*)$, and its throughput is $\tau_0(k^*)$.

Region k*0′: The cycle is $\theta_0'(k^*)$, and its throughput is $\tau_0'(k^*)$.

Region k*1: The cycle is $\theta_1(k^*)$, and its throughput is $\tau_1(k^*)$.

Region k*2: The cycle is $\theta_2(k^*)$, and its throughput is $\tau_2(k^*)$.

Region k*3: The cycle is $\theta_3(k^*)$, and its throughput is $\tau_3(k^*)$

Region k*4: The cycle is $\theta_4(k^*)$, and its throughput is $\tau_4(k^*)$.

Region k*5: The cycle is $\theta_5(k^*)$, and its throughput is $\tau_5(k^*)$.

Region k*6: The cycle is $\theta_6(k^*)$, and its throughput is $\tau_6(k^*)$.

Region k*7: The cycle is $\theta_7(k^*)$, and its throughput is $\tau_7(k^*)$.

Region k*8: The cycle is $\theta_8(k^*)$, and its throughput is $\tau_8(k^*)$.

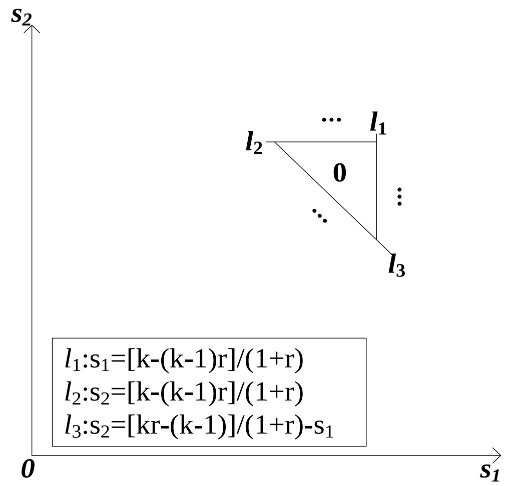
Region k*9: The cycle is $\theta_9(k^*)$, and its throughput is $\tau_9(k^*)$.

Region k*10: The cycle is $\theta_7(k^*)$ or $\theta_8(k^*)$, and its throughput is $\tau_7(k^*)$ or $\tau_8(k^*)$.

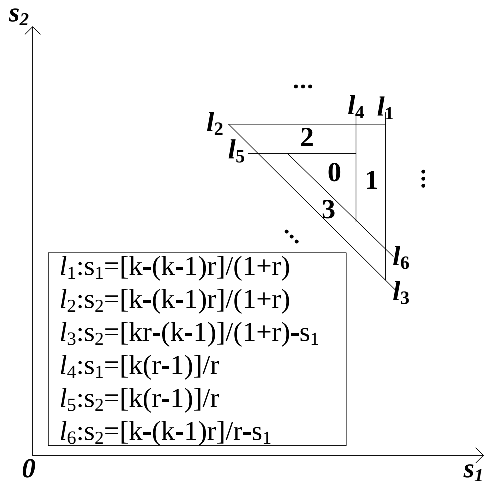
Region k*11: The cycle is $\theta_8(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_8(k^*)$ or $\tau_9(k^*)$.

Region k*12: The cycle is $\theta_7(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_7(k^*)$ or $\tau_9(k^*)$.

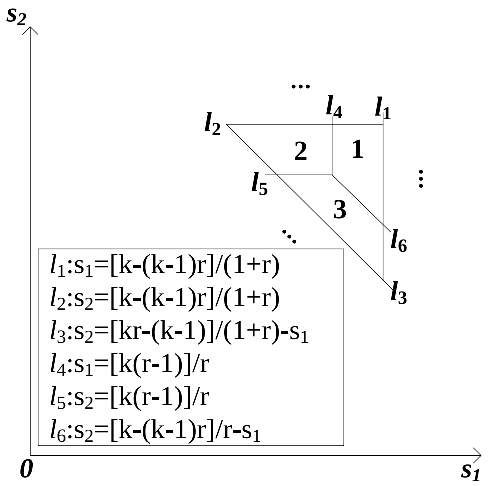
Region k*13: The cycle is $\theta_7(k^*), \theta_8(k^*)$ or $\theta_9(k^*)$, and its throughput is $\tau_7(k^*)$, $\tau_8(k^*)$ or $\tau_9(k^*)$. □



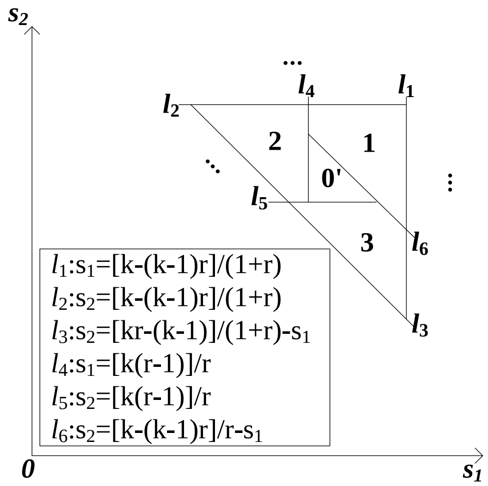

(a) $\Omega(k^*) \geq \Phi(k^*)$



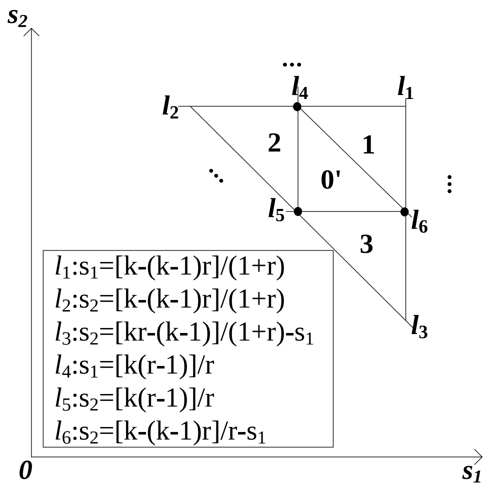

(b) $\Phi(k^*) > \Omega(k^*) > \frac{1}{3}$



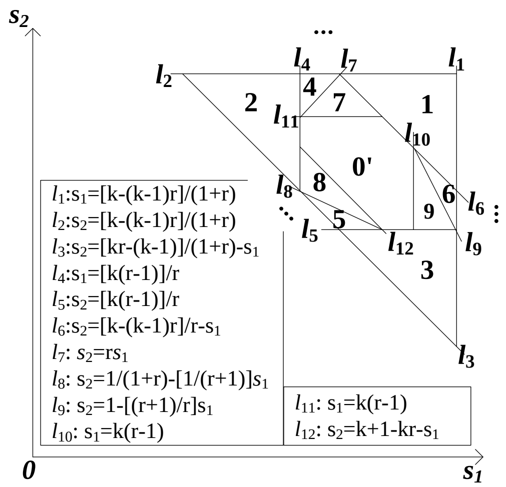

(c) $\Omega(k^*) = \frac{1}{3}$



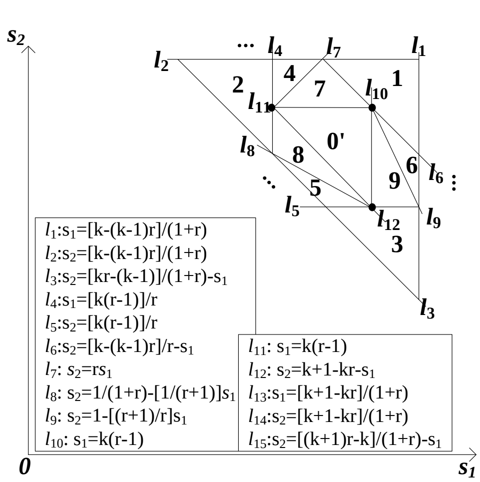

(d) $\frac{1}{3} > \Omega(k^*) > \frac{1}{r}\Phi(k^*)$



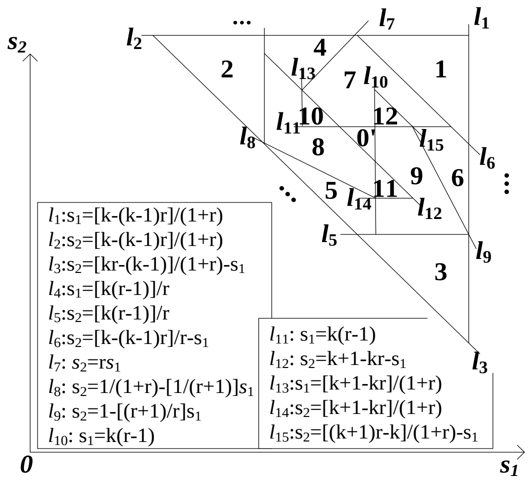

(e) $\Omega(k^*) = \frac{1}{r}\Phi(k^*)$



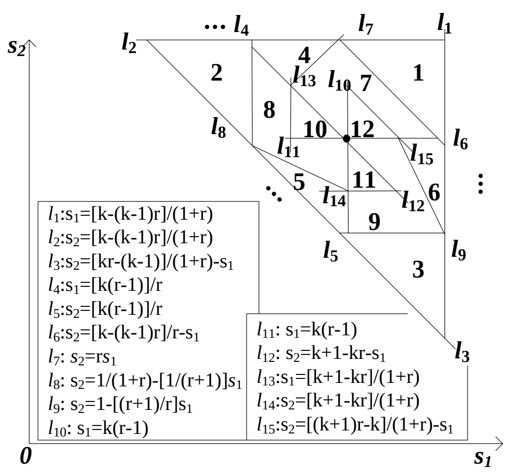

(f) $\frac{1}{r}\Phi(k^*) > \Omega(k^*) > \frac{r+1}{2r^2}\Phi(k^*)$

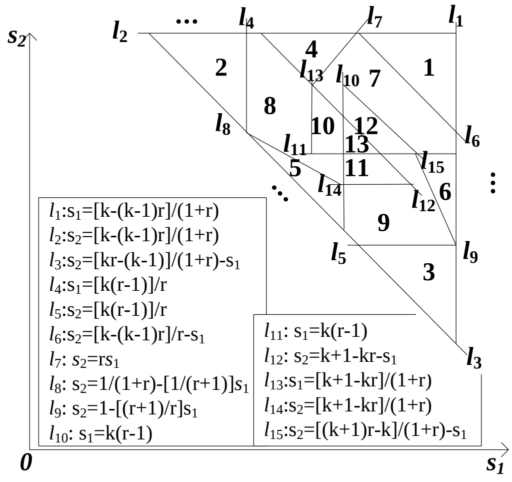



(g) $\Omega(k^*) = \frac{r+1}{2r^2}\Phi(k^*)$

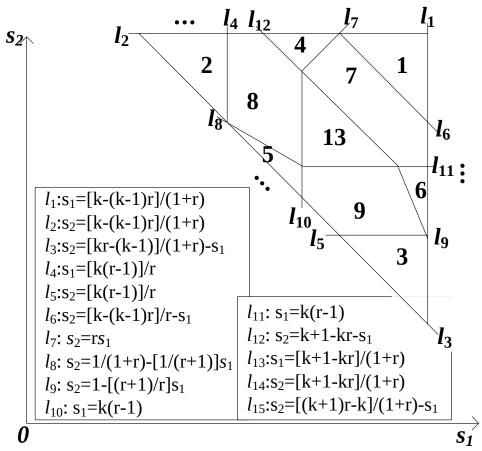



(h) $\frac{r+1}{2r^2}\Phi(k^*) > \Omega(k^*) > \frac{r+1}{2r}\Phi(k^*+1)$

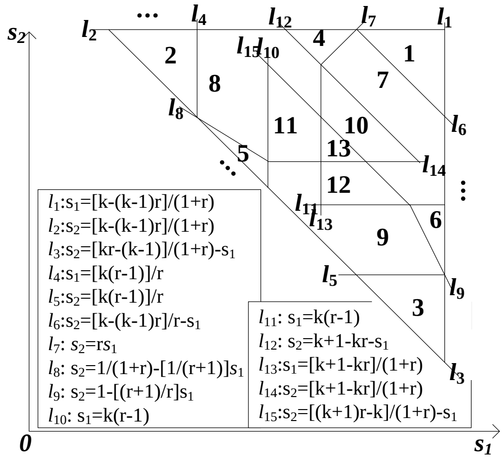


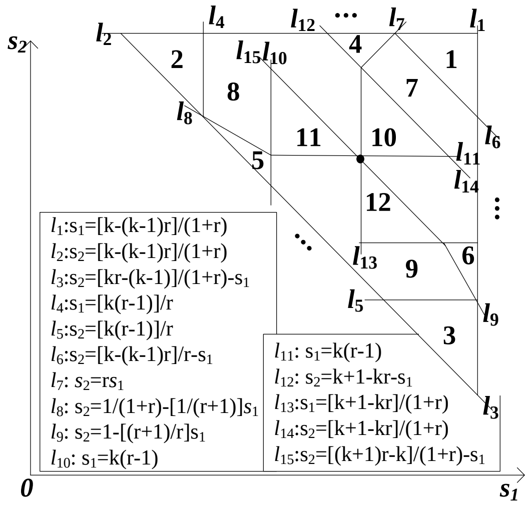

(i) $\Omega(k^*) = \frac{r+1}{2r}\Phi(k^*+1)$ (j) $\frac{r+1}{2r}\Phi(k^*+1) > \Omega(k^*) > \frac{1}{r}\ \Phi(k^*+1)$



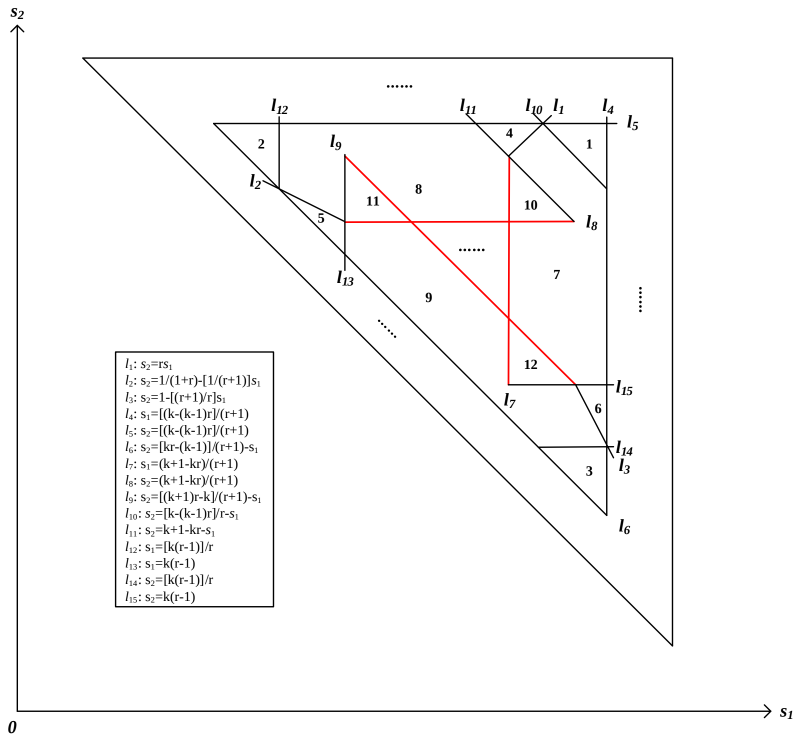

(k) $\Omega(k^*) = \frac{1}{r}\ \Phi(k^*+1)$




(l) $\frac{1}{r}\ \Phi(k^*+1) > \ \Omega(k^*) > \frac{2-r}{3r}$




(m) $\Omega(k^*) = \frac{2-r}{3r}$




(The red lines mean the maximal throughput)

Figure A.3. The evolution of 15 sub regions for $D(k^*)$.

✧ **<u>Proof of THEOREM 3.</u>**

Let $\delta_1 = \frac{k+1-kr}{r+1}$, $\delta_2 = k(r-1)$, $\delta_3 = \frac{r(k+1-kr)}{r+1}$, and $\delta_4 = \frac{(2k+1)(r-1)}{r+1}$.

There are three exclusive cases. Case (1): $a^{(0)} = b_1 \ or \ p_1$; Case (2): $a^{(0)} = b_2 \ or \ p_2$; Case (3): $a^{(0)} = b_3 \ or \ p_3$.

Case (1): $a^{(0)} = b_1 \ or \ p_1$.

(A) For region k1, the cycle is $\theta_1(k)$, and its throughput is $\tau_1(k)$. Given the velocity of worker 2 $v_2$, the throughput $\tau_1(k)$ increases as the work amount on station 3 $s_3$, increases. In region k1, when $s_3 = \delta_2/r$ ($l_{10}$ in Figure 8), the maximal throughput is achieved, which is $\frac{4r}{2k-k(r-1)/r} \times v_2 = \frac{4r}{r+1} \times v_2 < (r+1) \times v_2 = v_1 + v_2$.

(B) For region k2, the cycle is $\theta_2(k)$, and its throughput is $\tau_2(k)$. Given the velocity of worker 2 $v_2$, the throughput $\tau_2(k)$ increases as the work amount on station 1 $s_1$, increases. In region k1, when $s_1 = \delta_2/r$ ($l_{12}$ in Figure 8), the maximal throughput is achieved, which is $\frac{4r}{2k-k(r-1)/r} \times v_2 = \frac{4r}{r+1} \times v_2 < (r+1) \times v_2 = v_1 + v_2$.

(C) For region k3, the cycle is $\theta_3(k)$, and its throughput is $\tau_3(k)$. Given the velocity of worker 2 $v_2$, the throughput $\tau_3(k)$ increases as the work amount on station 2 $s_2$, increases. In region k3, when $s_2 = \delta_2/r$ ($l_{14}$ in Figure 8), the maximal throughput is achieved, which is $\frac{4r}{2k-k(r-1)/r} \times v_2 = \frac{4r}{r+1} \times v_2 < (r+1) \times v_2 = v_1 + v_2$.

(D) For region k4, the cycle is $\theta_4(k)$, and its throughput is $\tau_4(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_4(k)$ increases as the work amount on station 2 $s_2$, decreases. In region k4, because $s_2 < \delta_3$ ( the intersection of lines $l_{11}$ and $l_1$), $\tau_4(k) < \frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(E) For region k5, the cycle is $\theta_5(k)$, and its throughput is $\tau_5(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_5(k)$ increases as the work amount on station 3 $s_3$, decreases. In region k5, because $s_3 < \delta_3$( the intersection of lines $l_{13}$ and $l_2$), $\tau_5(k) < \frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(F) For region k6, the cycle is $\theta_6(k)$, and its throughput is $\tau_6(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_6(k)$ increases as the work amount on station 1 $s_1$, decreases. In region k6, because $s_1 < \frac{r(k+1-kr)}{r+1}$ ( the intersection of lines $l_{15}$ and $l_3$), $\tau_6(k) < \frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(G) For region k7, the cycle is $\theta_7(k)$, and its throughput is $\tau_7(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_7(k)$ increases as the work amount on station 1 $s_1$, decreases. In region k7, when $s_1 = \delta_1$ ($l_7$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(H) For region k8, the cycle is $\theta_8(k)$, and its throughput is $\tau_8(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_8(k)$ increases as the work amount on station 2 $s_2$, decreases. In region k8, when $s_2 = \delta_1$ ($l_8$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(I) For region k9, the cycle is $\theta_9(k)$, and its throughput is $\tau_9(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_9(k)$ increases as the work amount on station 3 $s_3$, decreases. In region k9, when $s_3 = \delta_1$ ($l_9$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(J) For region k10, the cycle is $\theta_7(k)$, and its throughput is $\tau_7(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_7(k)$ increases as the work amount on station 1 $s_1$, decreases. In region k10, when $s_1 = \delta_1$ ($l_7$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(K) For region k11, the cycle is $\theta_8(k)$, and its throughput is $\tau_8(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_8(k)$ increases as the work amount on station 2 $s_2$, decreases. In region k11, when $s_2 = \delta_1$ ($l_8$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

(L) For region k12, the cycle is $\theta_7(k)$, and its throughput is $\tau_7(k)$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput $\tau_7(k)$ increases as the work amount on station 1 $s_1$, decreases. In region k10, when $s_1 = \delta_1$ ($l_7$ in Figure 8), the maximal throughput is achieved, which is $\frac{2k+1}{k-(k+1-kr)/(r+1)} \times v_2 = v_1 + v_2$.

Combined with (A)-(L), if the initial iterate $a^{(0)} = b_1 \ or \ p_1$, when $s_1 = \delta_1$ $(\delta_2 \le s_2 \le \delta_3)$, $s_2 = \delta_1(\delta_2 \le s_1 \le \delta_1)$ or $s_3 = \delta_1(\delta_4 \le s_1 \le \delta_1)$, the maximal throughput is achieved, which is $v_1 + v_2$.

Case (2): $a^{(0)} = b_2 \ or \ p_2$.

Similarly, if the initial iterate $a^{(0)} = b_2 \ or \ p_2$, when $s_2 = \delta_1$ $(\delta_2 \le s_3 \le \delta_3)$, $s_3 = \delta_1(\delta_2 \le s_2 \le \delta_1)$ or $s_1 = \delta_1(\delta_4 \le s_2 \le \delta_1)$, the maximal throughput is achieved, which is $v_1 + v_2$.

Case (3): $a^{(0)} = b_3 \ or \ p_3$.

Similarly, if the initial iterate $a^{(0)} = b_3 \ or \ p_3$, when $s_3 = \delta_1$ $(\delta_2 \le s_1 \le \delta_3)$, $s_1 = \delta_1(\delta_2 \le s_3 \le \delta_1)$ or $s_2 = \delta_1(\delta_4 \le s_1 \le \delta_1)$, the maximal throughput is achieved, which is $v_1 + v_2$. □

✧ **Proof of LEMMA 8.**

The immediate positions of workers 1 and 2 after $p_i$ are the end and the start of station $i$, respectively.

First, we prove that if $r \ge 2$ and the cycle is $p_i(c_1c_2)^k c_1 p_i, i = 1,2,\dots,m$, then $s_i \ge 1/(r+1)$. Assume the contrary, i.e., $s_i < 1/(r+1)$. Since $s_i < \frac{1}{r+1} \Rightarrow \frac{1-s_i}{v_1} > \frac{s_i}{v_2}$, worker 2 departs from the end of station $i$ before worker 1 can complete an item and reach the start of station $i$. Due to $r \ge 2 \Leftrightarrow \frac{2}{v_1} \le \frac{1}{v_2}, b_j$ or $p_j$ $(j \ne i)$ will occur before worker 2 completes an item and reaches the start of station $i$. Thus, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed. So, we prove $s_i \ge 1/(r+1)$.

Next, we prove that $k = 0$. Since $s_i \ge \frac{1}{r+1} \Rightarrow \frac{1-s_i}{v_1} \le \frac{s_i}{v_2}$, worker 1 can reach the start of station $i$ before worker 2 finishes his work on station $i$. It means that worker 2 continues working on station $i$ and $c_2$ will never occur. So, we prove $k = 0$. □

✧ **Proof of THEOREM 4.**

The immediate positions of workers 1 and 2 after $p_i$ are the end and the start of station $i$, respectively. If $s_i \ge \frac{1}{r+1} \Rightarrow \frac{1-s_i}{v_1} \le \frac{s_i}{v_2}$, which implies that worker 1 can complete an item and reach the start of station $i$ before worker 2 finishes his work on station $i$. Thus, $p_i$ will occur. The orbit $p_i \to f(p_i): p_i \to \cdots$ or $b_i \to f(b_i): p_i \to \cdots$ is obtained. Therefore, the cycle $p_i c_1 p_i$ is constructed.

With the cycle of $p_i c_1 p_i$, its throughput is $v_2/s_i$. Given the velocity of worker 2 $v_2$, the throughput $v_2/s_i$ increases as the work amount on station $i$, $s_i$, decreases. When $s_i = \frac{1}{r+1}$, the maximal throughput is achieved, which is $(r+1) \times v_2 = v_1 + v_2$. □

✧ **Proof of LEMMA 9.**

The immediate positions of workers 1 and 2 after $p_i$ are the end and the start of station $i$, respectively.

(A) If $s_i \geq \frac{1}{r+1}$, then $k = 0$.

Because $s_i \geq \frac{1}{r+1} \Leftrightarrow \frac{s_i}{v_2} \geq \frac{1-s_i}{v_1}$, worker 1 can complete an item and return to the start of station $i$ before worker 2 finishes his work on station $i$. It means that worker 2 remains working on station $i$ and $c_2$ will never occur. Thus, we have $k = 0$.

(B) If $s_i < \frac{1}{r+1}$, then $1 \leq k \leq \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil$.

Suppose that $k < 1 \Rightarrow k = 0$ or $k > \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil \Rightarrow k \geq \frac{m-1-r}{m(r-1)} + 1$. We have two cases: (1) $k = 0$ and (2) $k \geq \frac{m-1-r}{m(r-1)} + 1$.

(1) $k = 0$

If $k = 0$, worker 2 remains working on station $i$ and $c_2$ will never occur. Thus, we have $\frac{s_i}{v_2} \geq \frac{1-s_i}{v_1} \Rightarrow s_i \geq \frac{1}{r+1}$, which contradicts $s_i < \frac{1}{r+1}$. So, we have $k \geq 1$.

(2) $k \geq \frac{m-1-r}{m(r-1)} + 1$

If $k \geq \frac{m-1-r}{m(r-1)} + 1$ , we have $\frac{k-1+1/m}{v_2} \geq \frac{k-1/m}{v_1}$. There are two sub cases: (a) $s_i \geq 1/m$ and (b) $s_i < 1/m$.

(a) $s_i \geq 1/m$

If $s_i \geq \frac{1}{m}$, we have $\frac{k-1+s_i}{v_2} \geq \frac{k-1+1/m}{v_2} \geq \frac{k-1/m}{v_1} \geq \frac{k-s_i}{v_1}$. It implies that the next iterate will occur before worker 1 completes $k$ items and then reaches the start of station $i$. Thus, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed

(b) $s_i < 1/m$

Assume that for any station $j, s_j < 1/m$. We have $\sum_{j=1}^{m} s_j < 1$. Therefore, if $s_i < 1/m$, there exists at least one station $j$ ($j \neq i$), $s_j \geq 1/m$. Since $s_j \geq 1/m$, we have $\frac{k-1+s_j}{v_2} \geq \frac{k-1+1/m}{v_2} \geq \frac{k-1/m}{v_1} \geq \frac{k-s_j}{v_1}$. There are two exclusive cases: (i) $j = i+1, \ldots, m$ and (ii) $j = 1, \ldots, i-1$.

(i) $j = i+1, \ldots, m$

The time for worker 1 to complete $k$ items and reach the start of station $j$ is $t_1 = \frac{s_{i+1}+\cdots+s_m+k-1+s_1+\cdots+s_{j-1}}{v_1} = \frac{k+s_{i+1}+\cdots+s_{j-1}}{v_1}$. The time for worker 2 to complete *k-1* items and reach the end of station $j$ is $t_2 = \frac{s_i+\cdots+s_m+k-2+s_1+\cdots+s_j}{v_2} = \frac{k-1+s_i+\cdots+s_j}{v_2}$. Combined with $\frac{k-1+s_j}{v_2} \geq \frac{k-s_j}{v_1}$, we have $t_2 \geq \frac{k-s_j}{v_1} + \frac{s_i+\cdots+s_{j-1}}{v_2} = t_1 + \frac{s_i+\cdots+s_{j-1}}{v_2} - \frac{s_{i+1}+\cdots+s_j}{v_1}$.

If $\frac{s_i+\cdots+s_{j-1}}{v_2} < \frac{s_{i+1}+\cdots+s_j}{v_1}$, worker 2 can reach the start of station $j$ before worker 1 finishes his work on station $j$. We have $f(p_i) = b_j$. Thus, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed.

If $\frac{s_i+\cdots+s_{j-1}}{v_2} \geq \frac{s_{i+1}+\cdots+s_j}{v_1}$, we have $t_2 \geq t_1$. The next iterate will occur before worker 1 completes $k$ items and reaches the start of station $j$. Also, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed.

(ii) $j = 1, \ldots, i-1$.

The time for worker 1 to complete $k+1$ items and reach the start of station $j$ is $t_1 = \frac{s_{i+1}+\cdots+s_m+k+s_1+\cdots+s_{j-1}}{v_1} = \frac{k+1-(s_j+\cdots+s_i)}{v_1}$. The time for worker 2 to complete $k$ items and reach the end of station $j$ is $t_2 = \frac{s_i+\cdots+s_m+k-1+s_1+\cdots+s_j}{v_2} = \frac{k-(s_{j+1}+\cdots+s_{i-1})}{v_2}$. Combined with $\frac{k-1+s_j}{v_2} \geq \frac{k-s_j}{v_1}$, we have

$$t_2 \geq \frac{k-s_j}{v_1} + \frac{1-(s_j+\cdots+s_{i-1})}{v_2} = t_1 + \frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2} - \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1}.$$

If $\frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2} < \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1}$, worker 2 can reach the start of station $j$ before worker 1 finishes his work on station $j$. We have $f(p_i) = b_j$. Thus, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed.

If $\frac{(s_i+\cdots+s_m)+(s_1+\cdots+s_{j-1})}{v_2} \geq \frac{(s_{i+1}+\cdots+s_m)+(s_1+\cdots+s_j)}{v_1}$, we have $t_2 \geq t_1$. The next iterate will occur before worker 1 completes $k+1$ items and reaches the start of station $j$. Also, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed.

Taken together case (1) and case (2), under the assumptions of $k = 0$ and $k \geq$

$\frac{m-1-r}{m(r-1)}+1$, the cycle $p_i(c_1c_2)^k c_1 p_i$ cannot be constructed. So, we can conclude that if $s_i < \frac{1}{r+1}$, then $1 \le k \le \left\lceil \frac{m-1-r}{m(r-1)} \right\rceil$. □

✧ **Proof of THEOREM 5.**

The immediate positions of workers 1 and 2 after $p_i$ or $b_i$ are the end and the start of station $i$, respectively.

(A) $k=0$

If $s_i \ge \frac{1}{r+1} \Rightarrow \frac{1-s_i}{v_1} \le \frac{s_i}{v_2}$, which implies that worker 1 can complete an item and reach the start of station $i$ before worker 2 finishes his worker on station $i$. Thus, $p_i$ will occur. The orbit $p_i \to f(p_i): p_i \to \cdots$ or $b_i \to f(b_i): p_i \to \cdots$ is obtained. Therefore, the cycle $p_i c_1 p_i$ is constructed.

With the cycle $p_i c_1 p_i$, its throughput is $v_2/s_i$. Given the velocity of worker 2 $v_2$, the throughput $v_2/s_i$ increases as the work amount on station $i$, $s_i$, decreases.

When $s_i = \frac{1}{r+1}$, the maximal throughput is achieved, which is $(r+1)\times v_2 = v_1 + v_2$.

(B) $k = 1,2,\dots,k^*$

First, we prove that blocking will not occur.

Based on condition (2), for $j = i+1,\dots,m$, we have $rs_i + (r-1)\sum_{o=i+1}^{j-1} s_o \ge s_j \Rightarrow \frac{s_i+\sum_{o=i+1}^{j-1} s_o}{v_2} \ge \frac{\sum_{o=i+1}^{j-1} s_o + s_j}{v_1}$; for $j = 1,\dots,i-1$, we have $rs_i + (r-1)\sum_{o=i+1}^{m} s_o + (r-1)\sum_{o=1}^{j-1} s_o \ge s_j \Rightarrow \frac{s_i+\sum_{o=i+1}^{m} s_o+\sum_{o=1}^{j-1} s_o}{v_2} \ge \frac{\sum_{o=i+1}^{m} s_o+\sum_{o=1}^{j-1} s_o+s_j}{v_1}$. It means that for any $j\ (j \ne i)$, worker 1 has departed from the end of station $j$ before worker 2 reaches the start of station $j$. So, any $b_j (j \ne i)$ will not occur.

Next, we prove that passing will not occur before worker 1 completes $k$ items and then reaches the start of station $i$.

Assume the contrary, let $j$ be the index of that station where passing occurs before worker 1 completes $k$ items and then reaches the start of station $i$. The time for worker 1 to complete $k'(k' \le k)$ items and then reach the start of station $j$ is $t_1 = \frac{s_{i+1}+\cdots+s_m+k'-1+s_1+\cdots+s_{j-1}}{v_1}$. The time for worker 2 to complete $k'-1$ items and then reach the end of station $j$ is $t_2 = \frac{s_i+\cdots+s_m+k'-2+s_1+\cdots+s_j}{v_2}$. Based on condition (3), we have $(r+1)s_i < k+r-kr$ and $(r+1)s_j < k+r-kr$. Thus, $s_i + s_j < \frac{2(k+r-kr)}{r+1} < k-(k-1)r$. Since we assume that passing occurs on station $j$, we have

$t_1 \leq t_2$. There are two exclusive cases: (a) $j = 1,2,\dots,i-1$ and (b) $j = i+1,\dots,m$.

Case (a): $j = 1,\dots,i-1$.

Since $j = 1,\dots,i-1$, we have $k' \leq k$. Because $t_1 \leq t_2$, we have $\frac{k'-s_i-s_j-(s_{j+1}+\cdots+s_{i-1})}{v_1} \leq \frac{k'-1-(s_{j+1}+\cdots+s_{i-1})}{v_2}$. Combined with $v_1 > v_2 \Leftrightarrow \frac{(s_{j+1}+\cdots+s_{i-1})}{v_1} < \frac{(s_{j+1}+\cdots+s_{i-1})}{v_2}$, we have $\frac{k'-s_i-s_j}{v_1} < \frac{k'-1}{v_2} \Rightarrow s_i + s_j > k' - (k'-1)r \geq k - (k-1)r$ due to $k' \leq k$, which contradicts $s_i + s_j < k - (k-1)r$.

Case (b): $j = i+1,\dots,m$.

Since $j = i+1,\dots,m$, we have $k' \leq k-1$. Because $t_1 \leq t_2$, we have $\frac{k'+1-s_i-s_j-(s_1+\cdots+s_{i-1})-(s_{j+1}+\cdots+s_m)}{v_1} \leq \frac{k'-(s_1+\cdots+s_{i-1})-(s_{j+1}+s_m)}{v_2}$. Combined with $v_1 > v_2 \Leftrightarrow \frac{(s_1+\cdots+s_{i-1})+(s_{j+1}+\cdots+s_m)}{v_1} < \frac{(s_1+\cdots+s_{i-1})+(s_{j+1}+\cdots+s_m)}{v_2}$, we have $\frac{k'}{v_2} > \frac{k'+1-s_i-s_j}{v_1} \Rightarrow s_i + s_j > (1-r)k' + 1 \geq (1-r)(k-1) + 1 = k - (k-1)r$ due to $k' \leq k-1$, again contradicting $s_i + s_j < k - (k-1)r$. Note that, it is impossible for $k' = k$ since $j > i$ in Case (b).

Third, we prove that passing will not occur before worker 1 completes $k+1$ items and reaches the start of station $i$.

Based on condition (3), since $(r+1)s_i < k + r - kr \Leftrightarrow \frac{k-s_i}{v_1} > \frac{k-1+s_i}{v_2}$, when worker 1 completes $k$ items and then reaches the start of station $i$, worker 2 has completed $k-1$ items and departed from station $i$.

Based on condition (2), for $j = i+1,\dots,m$, we have $\frac{\sum_{o=i}^{m} s_o + k - 2 + \sum_{o=1}^{j} s_o}{v_2} < \frac{\sum_{o=i+1}^{m} s_o + k - 1 + \sum_{o=1}^{j-1} s_o}{v_1}$. It means that worker 2 has departed from the end of station $j$ when worker 1 can complete k items and then reach the start of station $j$. For $j = 1,\dots,i-1$, we have $\frac{\sum_{o=i}^{m} s_o + k - 1 + \sum_{o=1}^{j} s_o}{v_2} < \frac{\sum_{o=i+1}^{m} s_o + k + \sum_{o=1}^{j-1} s_o}{v_1}$. It means that worker 2 has departed from the end of station $j$ when worker 1 can complete $k+1$ items and reach the start of station $j$. So, any $p_j (j \neq i)$ will not occur.

Finally, we prove that passing will occur when worker 1 completes $k+1$ items and reaches the start of station $i$.

Based on condition (3), since $(r+1)s_i \geq k + 1 - kr$, we have $\frac{k+s_i}{v_2} \geq \frac{k+1-s_i}{v_1}$. It means that worker 1 can complete $k+1$ items and reach the start of station $i$ before worker 2 completes $k$ items and departs from the end of station $i$. Thus, $p_i$ will occur.

Thus, the orbit $p_i \to f(p_i): p_i \to \cdots$ or $b_i \to f(b_i): p_i \to \cdots$ is obtained. The cycle $p_i(c_1c_2)^k c_1 p_i$ is constructed.

With the cycle $p_i(c_1c_2)^k c_1 p_i$, its throughput is $\frac{2k+1}{k+s_i} \times v_2$. Given the velocities of the two workers $v_1$ and $v_2$, the throughput increases as the work amount on station $i$, $s_i$, decreases.

When $s_i = \frac{k+1-kr}{r+1}$ the maximal throughput is achieved, which is $(r+1) \times v_2 = v_1 + v_2$. □

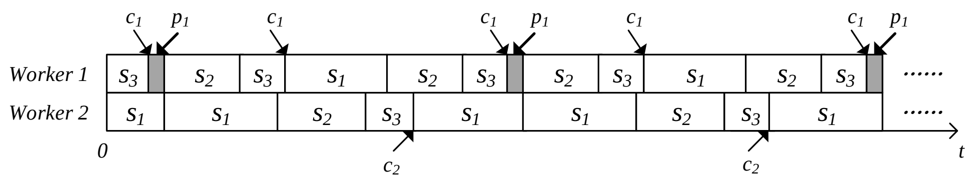
**A.2. Illustrative examples**

Example 1:

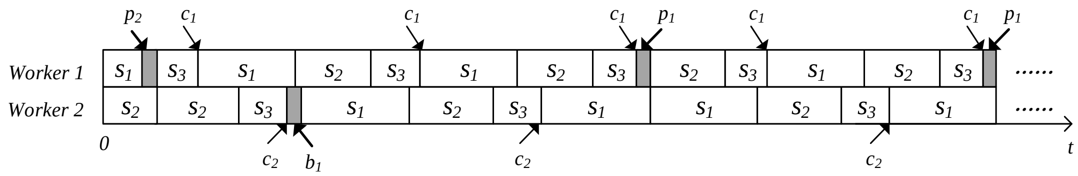
Consider a rotating *seru* with two workers and three stations. The work velocities of the workers are $v_1 = 2.0$ and $v_2 = 1.8$, and the work content of each station is $s_1 = 0.45$, $s_2 = 0.35$, and $s_3 = 0.20$. Figures A.4(a) and A.4(b) show the period-1 cycles of this *seru*. In the figures, the left and right borders of a station box indicate the start and finish times of a worker on that station. A grey box represents a worker who is idle due to passing or blocking behaviors.

In Figure A.4(a), the initial positions of the two workers are represented by $x_1 = 0.9$ and $x_2 = 0.2$. The time it takes for worker 1 to reach the start of station 1 is $t_1 = \frac{1-x_1}{v_1} = 0.05$, while the time it takes for worker 2 to reach the end of station 1 is $t_2 = \frac{0.45-x_2}{v_2} = 0.14$. Since $t_1 < t_2$, when worker 1 arrives at the start of station 1, that station is occupied by worker 2. Thus, we have $a^{(0)} = p_1$. The positions of worker 1 and worker 2 immediately after $p_1$ are the end and start of station 1, respectively. The time it takes for worker 1 to complete two items and reach the start of station 1 again is $t_1' = \frac{0.35+0.2+1}{2.0} = 0.775$ and the time it takes for worker 2 to complete one item and reach the end of station 1 is $t_2' = \frac{1+0.45}{1.8} = 0.806$. Since $t_1' < t_2'$, we have $a^{(1)} = p_1$, and an orbit $p_1 \to f(p_1): p_1 \to \cdots f(p_1): p_1 \to \cdots$ (or $\ldots p_1 \to p_1 \ldots$) is obtained.

In Figure A.4(b), $x_1 = 0.4$ and $x_2 = 0.5$. As with Figure 4(a), we have $a^{(0)} = p_2$, $a^{(1)} = b_1$, $a^{(2)} = p_1$ and $a^{(3)} = p_1$, and an orbit $p_2 \to f(p_2): b_1 \to f(b_1): p_1 \to f(p_1): p_1 \to \cdots f(p_1): p_1 \to \cdots$ (or $\ldots p_1 \to p_1 \ldots$) is obtained.



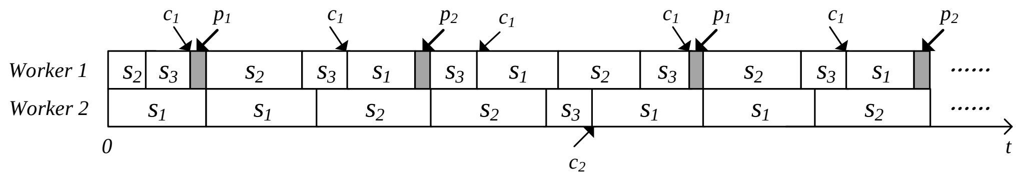

(a) $a^{(0)} = p_1$ and $s_1 = 0.45$, $s_2 = 0.35$, $s_3 = 0.20$



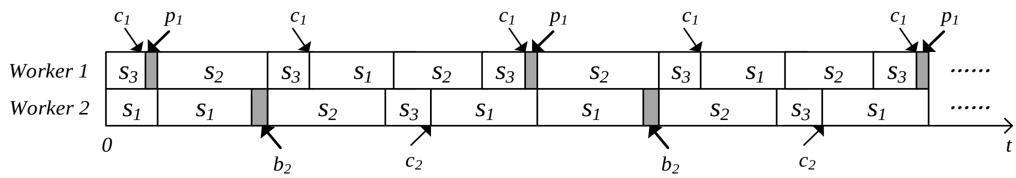

(b) $a^{(0)} = p_2$ and $s_1 = 0.45,\ s_2 = 0.35,\ s_3 = 0.20$

Figure A.4. Period-1 cycles for a two-worker, three-station rotating *seru*.

In Figure A.5 (a) and (b), period-2 cycles can be constructed. We set $x_1 = 0.9$ and $x_2 = 0.1$. For Figure A.5(a), we set $s_1 = 0.45,\ s_2 = 0.46$ and $s_3 = 0.09$. The time it takes for worker 1 to arrive at the start of station 1 is $t_1 = \frac{1-0.9}{2.0} = 0.05$, while the time for worker 2 to arrive at the end of station 1 is $t_2 = \frac{0.45-0.1}{1.8} = 0.19$. Since $t_1 < t_2$, we have $a^{(0)} = p_1$. The time for worker 1 to complete an item and then arrive at the start of station 2 is $t_1^{'} = \frac{1}{2.0} = 0.500$, and the time for worker 2 to compete his work on station 1 and station 2 is $t_2^{'} = \frac{0.45+0.46}{1.8} = 0.506$. Since $t_1^{'} < t_2^{'}$, we have $a^{(1)} = p_2$. We continue to track the dynamics of the two workers, we have $a^{(2)} = p_1$, and the orbit $p_1 \to f(p_1){:}\, p_2 \to f(p_2){:}\, p_1 \to \cdots$ (or $\dots p_1 \to p_2 \to p_1 \to p_2 \dots$) is obtained. For Figure A.5(b), we set $s_1 = 0.42,\ s_2 = 0.47,\ s_3 = 0.11$. Due to the larger work content at station 2, even with the same initial iterate, we have $a^{(1)} = b_2,\ a^{(2)} = p_1$, and the orbit $p_1 \to f(p_1){:}\, b_2 \to f(b_2){:}\, p_1 \to \cdots$ (or $\dots p_1 \to b_2 \to p_1 \to b_2 \dots$) is obtained.




(a) $a^{(0)} = p_1$ and $s_1 = 0.45,\ s_2 = 0.46,\ s_3 = 0.09$




(b) $a^{(0)} = p_1$ and $s_1 = 0.42,\ s_2 = 0.47,\ s_3 = 0.11$

Figure A.5. Period-2 cycles for a two-worker and three-station rotating *seru*.

<u>Example 2 (continued from Example 1):</u>

Consider Figure A.4 again. For Figure A.4(a), we have $a^{(0)} = p_1$ and $a^{(1)} = p_1$. Between $a^{(0)}$ and $a^{(1)}$, two items are completed by worker 1 and one item is completed by worker 2, so we have $k = 1$, and the behaviors between them are denoted as $p_1c_1c_2c_1p_1$ (i.e., (1) of Lemma 2). For Figure A.4(b), we have $a^{(0)} = p_2$, $a^{(1)} = b_1$, and $a^{(2)} = p_1$. Between $a^{(0)}$ and $a^{(1)}$, worker 1 and worker 2 complete one item each, so we have $k = 1$, and the behaviors between them are denoted as $p_2c_1c_2b_1$ (i.e., (2) of Lemma 2). Between $a^{(1)}$ and $a^{(2)}$, two items are completed by worker 1 and one item is completed by worker 2, so we have $k = 1$ and the behaviors between them are denoted as $b_1c_1c_2c_1p_1$ (i.e., (1) of Lemma 2).

According to (1) of Lemma 2, when $v_1 = 2.0$, $v_2 = 1.8$, and $m = 3$, we have $k \leq 4$. We illustrate the cases of $k = 2,3,4$ by changing the work content on the stations.

When the difference between the work content on different stations is relatively small, worker 2 has the opportunity to complete more items between two successive passing behaviors, resulting in a larger $k$. For example, given $s_1 = 0.40$, $s_2 = 0.35$, $s_3 = 0.25$, and starting with $a^{(0)} = p_1$, by analyzing the dynamics of the two workers, we can see that $a^{(1)} = p_1$. Between $a^{(0)}$ and $a^{(1)}$, three items are completed by worker 1 and two items are completed by worker 2. Thus, we have $k = 2$ and the behaviors between them can be denoted as $p_1c_1c_2c_1c_2c_1p_1$.

By adjusting the work content on the stations to $s_1 = 0.35$, $s_2 = 0.30$, and $s_3 = 0.35$, if $a^{(0)} = p_1$, we still have $a^{(1)} = p_1$. However, between $a^{(0)}$ and $a^{(1)}$, four items are completed by worker 1 and three items are completed by worker 2. Thus, we have $k = 3$ and the behaviors between them can be denoted as $p_1c_1c_2c_1c_2c_1c_2c_1p_1$.

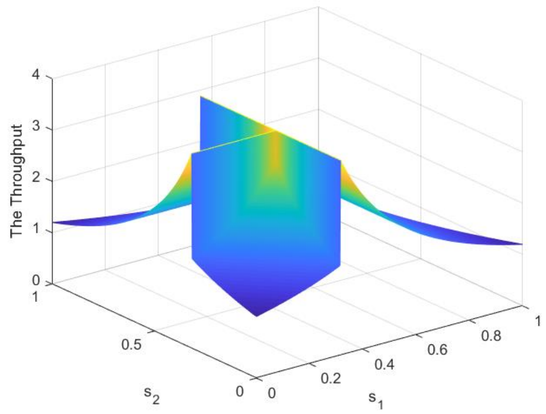
Given $s_1 = 0.31$, $s_2 = 0.34$, $s_3 = 0.35$ and $a^{(0)} = p_1$, the time it takes for worker 1 to complete five items and reach the start of station 1 is $t_1 = \frac{0.34+0.35+4}{2.0} = \frac{4.69}{2.0} = 2.35$, while the time it takes for worker 2 to complete four items and reach the end of station 1 is $t_2 = \frac{4+0.31}{2.0} = 2.16$. Since $t_1 > t_2$, $p_1$ will not occur. Continuing to analyze the dynamics of the workers, the time it takes for worker 1 to complete five items and reach the start of station 2 is $t_1' = \frac{0.34+0.35+4+0.31}{2.0} = 2.5$, while the time it takes for worker 2 to complete four items and reach the end of station 2 is $t_2' = \frac{4+0.31+0.34}{1.8} = 2.58$. Since $t_1' < t_2'$, we have $a^{(1)} = p_2$. Therefore, we have $k = 4$ and the behaviors between them can be denoted as $p_1c_1c_2c_1c_2c_1c_2c_1c_2c_1p_2$.

<u>Example 3:</u>

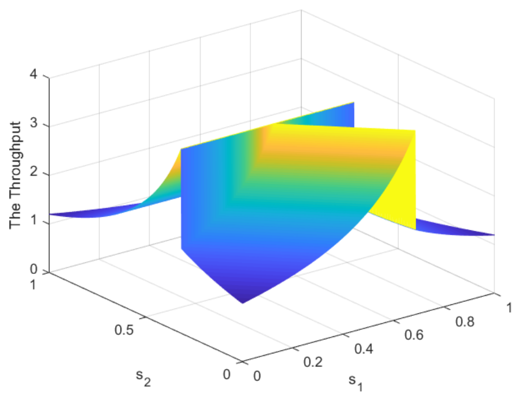
Given $v_1 = 2.6$ and $v_2 = 1.2$, we have $r = v_1/v_2 = 2.17$. Let $s_1 = 0.2$, $s_2 = 0.6$, $s_3 = 0.2$, and $a^{(0)} = p_1$. According to Lemma 3, we have $a^{(1)} = b_2$ because $rs_1 < s_2$, and $a^{(2)} = p_2$ because $s_2 \geq 1/(r+1)$. So, the orbit $p_1 \to f(p_1){:}\, b_2 \to f(b_2){:}\, p_2 \to f(p_2){:}\, p_2 \ldots$ is obtained. According to Lemma 4, the point $(s_1, s_2)$ is in region 4 (see Figure 4(a)), the cycle is $\theta_2 = p_2c_1p_2$ and the throughput is $\tau_2 = v_2/s_2 = 2.0$.

<u>Example 4 (continued from Example 3):</u>

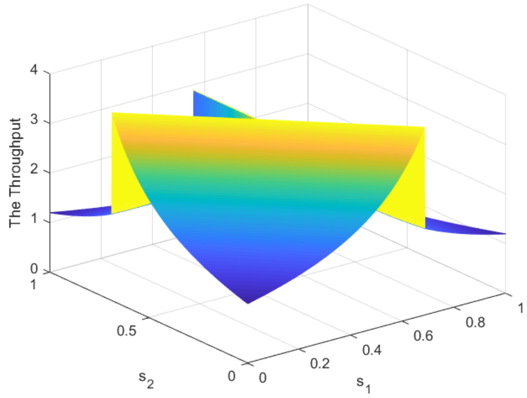
Again, $v_1 = 2.6$ and $v_2 = 1.2$. Figure A.6 shows the throughput in each region with different initial states. As the velocity ratio is $r \geq 2$, blocking will never occur and worker 2 will always be working on the station that has the largest work content among the three stations. Although worker 2 remains occupied at a single station, worker 1 may have to wait at the start of the station where passing occurs. The throughput of the system depends on worker 2's velocity and the amount of work content on the station where passing occurs. The throughput decreases monotonically with respect to worker 1's waiting time. Regions 4-7 have different throughputs, as shown in Figure A.6 (a)-(c). This is because they have multiple feasible cycles with different initial states. It is important to note that, according to Theorem 2, the maximal throughput $v_1 + v_2 = 3.8$ is achieved when the work content is distributed optimally among stations. If $a^{(0)} \in \{p_1, b_1\}$, the maximal throughput is achieved when $s_1 = 1/(r+1) = 0.32$; or when $s_2 = 0.32$ and $0 < s_1 < 0.32$. See Figure A.6(a). If $a^{(0)} \in \{p_2, b_2\}$, the maximal throughput is achieved when $s_2 = 0.32$; or when $s_3 = 0.32$ and $0 < s_2 < 0.32$. See Figure A.6(b). If $a^{(0)} \in \{p_3, b_3\}$, the maximal throughput is achieved when $s_3 = 0.32$; or when $s_1 = 0.32$ and $0 < s_3 < 0.32$. See Figure A.6(c).




(a) $p_1$ or $b_1$




(b) $p_2$ or $b_2$




(c) $p_3$ or $b_3$

Figure A.6. The throughput with different initial states when $v_1 = 2.6$ and $v_2 = 1.2$.

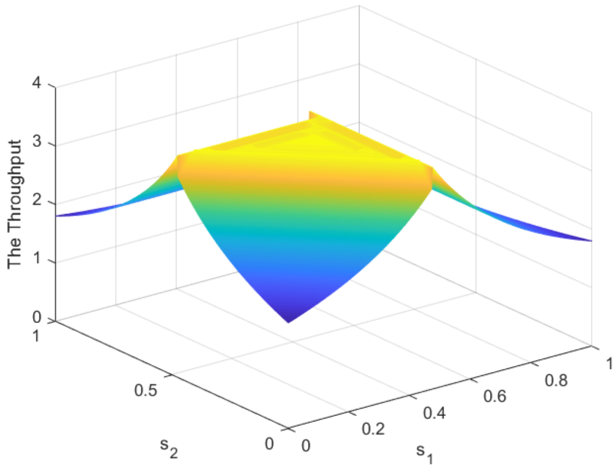
Example 5:

Given $v_1 = 2.0$ and $v_2 = 1.8$, we have $r = v_1/v_2 = 1.11$ and $k^* = 3$. Let $s_1 = 0.4$, $s_2 = 0.41$, $s_3 = 0.19$, and the initial state $a^{(0)} = p_1$. Because $T(2) = 0.4218$, $T(3) = 0.3697$, and $T(3) \leq (s_1, s_2) < T(2)$, the point $(s_1, s_2)$ is on the second layer $D(2)$. According to Lemma 5, as $a^{(0)} = p_1$, we have $a^{(1)} = p_2$ because of $rs_1 \geq s_2$, $rs_1 + (r-1)s_2 \geq s_3$, $s_3 \leq \Omega(k)$, and $a^{(2)} = p_1$ because of $rs_2 \geq s_3$, $rs_2 + (r-1)s_3 \geq s_1$, $s_1 > \Omega(k)$, $s_3 \leq r\Omega(k)$. Correspondingly, the orbit $p_1 \to f(p_1){:}\, p_2 \to f(p_2){:}\, p_1 \ldots$ is obtained. According to Lemma 6, the point $(s_1, s_2)$ is on sub region 1 of layer $D(2)$. The cycle is $\theta_1(2) = p_1 c_1 c_2 c_1 p_2 c_1 c_2 c_1 c_2 c_1 p_1$ and the throughput is $\tau_1(2) = \frac{8}{4-0.19} \times 1.8 = 3.78$.

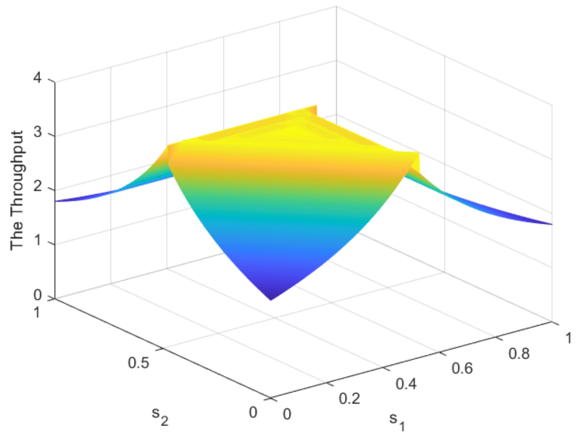
Example 6 (continued from Example 5).

Figure A.7 shows the throughput in each region for different initial states. The velocities are $v_1 = 2.0$ and $v_2 = 1.8$. In Regions 1-6, the throughput decreases monotonically because the three stations are less balanced, and worker 2 is kept busy on a single station. In contrast, the throughput remains stable and high for Region $D$ since the waiting times of worker 1 are not long due to the work content being more evenly distributed than in Regions 1-6. In Region $D$, worker 2 does not work on a single station, having opportunities to complete more items in cycles. Only small throughput differences exist in Regions 4-6, as shown in Figure A.7 (a)-(c), because the areas of Regions 4-6 become small with the decrease of the worker velocity ratio. By Theorem 3, the maximal throughput $v_1 + v_2 = 3.8$ is achieved if the work content is distributed optimally. Let us take layer $D(2)$ as an example. If $a^{(0)} \in \{p_1, b_1\}$, the maximal throughput is achieved when $s_1 = 0.37$ and $0.22 \leq s_2 \leq 0.41$, or when $s_2 = 0.37$ and $0.22 \leq s_1 \leq 0.37$, or when $s_3 = 0.37$ and $0.26 \leq s_1 \leq 0.37$, as in Figure A.7(a). If $a^{(0)} \in \{p_2, b_2\}$, the maximal throughput is achieved when $s_2 = 0.37$ and $0.22 \leq s_3 \leq$

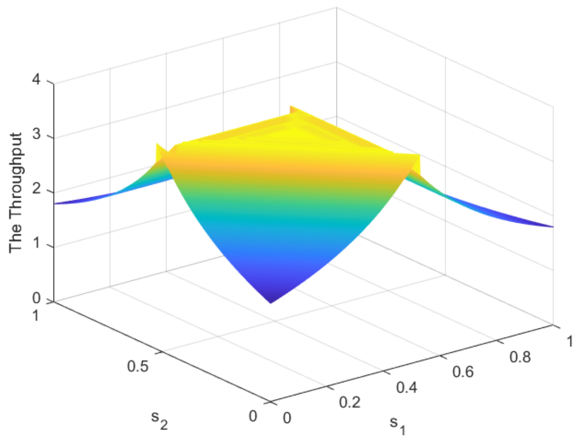
0.41, or when $s_3 = 0.37$ and $0.22 \leq s_2 \leq 0.37$, or when $s_1 = 0.37$ and $0.26 \leq s_2 \leq 0.37$, as in Figure A.7(b). If $a^{(0)} \in \{p_3, b_3\}$, the maximal throughput is achieved when $s_3 = 0.37$ and $0.22 \leq s_1 \leq 0.41$, or when $s_1 = 0.37$ and $0.22 \leq s_3 \leq 0.37$, or when $s_2 = 0.37$ and $0.26 \leq s_1 \leq 0.37$, as in Figure A.7(c).




(a) $p_1$ or $b_1$




(b) $p_2$ or $b_2$




(c) $p_3$ or $b_3$

Figure A.7. The throughput with different initial states when $v_1 = 2.0$ and $v_2 = 1.8$.

Example 7:

Consider a rotating *seru* of two workers and six stations, where $v_1 = 2.0$, $v_2 = 1.8$, and $r = v_1/v_2 = 1.11$. Let $s_1$ be the fixed station where passings occur. We have a cycle $p_1 \underbrace{c_1 c_2 \dots c_1 c_2}_{k} c_1 p_1$.

We set $s_1 = 0.5 \geq 1/(r+1)$ (Case (A) of Lemma 9) and $s_2 = s_3 = s_4 = s_5 = s_6 = 0.1$. The immediate positions of workers 1 and 2 after $p_1$ are the end and the start of station 1, respectively. Because $s_1 = 0.5 \geq \frac{1}{r+1} \Leftrightarrow \frac{s_1}{v_2} \geq \frac{1-s_1}{v_1}$, worker 1 can complete one item and return to the start of station 1 before worker 2 finishes her/his work on station 1. This means that worker 2 keeps working on station 1 and $c_2$ will never occur. Thus, we have $k = 0$.

Let us set $s_1 = 0.45 < 1/(r+1)$ (Case (B) of Lemma 9) and $s_2 = s_3 = s_4 = s_5 = s_6 = 0.11$. Because $s_1 = 0.45 \Leftrightarrow \frac{2-s_1}{v_1} < \frac{1+s_1}{v_2}$, before worker 2 completes one item and reaches the end of station 1, worker 1 completes two items and arrives at the start of station 1. In this case, we have $k = 1$.

According to (B) of Lemma 9, $k \leq k^* = \left\lceil \frac{\boldsymbol{m-1-r}}{\boldsymbol{m(r-1)}} \right\rceil = \lceil 5.89 \rceil = 6$. We can adjust the work content on the stations to $s_1 = 0.165$ and $s_2 = s_3 = s_4 = s_5 = s_6 = 0.167$. Since $s_1 = 0.165 \Leftrightarrow \frac{7-s_1}{v_1} < \frac{6+s_1}{v_2}$, worker 1 completes seven items and arrives at the start of station 1 before worker 1 completes six items and reaches the end of station 1. Thus, we have $k = 6$.

Example 8:

Consider a rotating *seru* of two workers and five stations, where $v_1 = 2.5$, $v_2 = 2.0$, and $r = v_1/v_2 = 1.25$.

We set $s_1 = 0.5$ (Case (A) of Theorem 5) and $s_2 = s_3 = s_4 = s_5 = 0.125$. Given the initial state $a^{(0)} = p_1$ or $b_1$, the immediate positions of workers 1 and 2 after $p_1$ or $b_1$ are the end and the start of station 1, respectively. Because $s_1 = 0.5 \geq \frac{1}{r+1} \Leftrightarrow \frac{s_1}{v_2} \geq \frac{1-s_1}{v_1}$ (Condition (1) of Theorem 5), we have $a^{(1)} = p_1$. Thus, the cycle $p_1(c_1 c_2)^k c_1 p_1$ is constructed, where $k = 0$. Furthermore, when $s_1 = \frac{1}{r+1} = 0.44 \Leftrightarrow \frac{s_1}{v_2} = \frac{1-s_1}{v_1}$, there is no waiting time for worker 1, and worker 2 keeps busy on station 1. Thus, the maximal throughput can be obtained, that is $v_1 + v_2 = 4.5$.

Let us set $s_3 = 0.3$ (Case (B) of Theorem 5), $s_1 = s_2 = s_4 = s_5 = 0.175$ and the initial state $a^{(0)} = p_3$ or $b_3$. The immediate positions of workers 1 and 2 after $p_3$ or $b_3$ are the end and the start of station 3, respectively. Since $s_3 = 0.3 < \frac{1}{r+1} \Leftrightarrow \frac{s_3}{v_2} < \frac{1-s_3}{v_1}$, worker 2 has departed from the end of station 3 before worker 1 has completed one item and returned to the start of station 3. Because $\frac{s_3}{v_2} \geq \frac{s_4}{v_1}$, $\frac{s_3+s_4}{v_2} \geq \frac{s_4+s_5}{v_1}$, $\frac{s_3+s_4+s_5}{v_2} \geq \frac{s_4+s_5+s_1}{v_1}$, and $\frac{s_3+s_4+s_5+s_1}{v_2} \geq \frac{s_4+s_5+s_1+s_2}{v_1}$ ($rs_i + (r-1)\sum_{o=i+1}^{j-1} s_o \geq s_j$ for $i = 3$ and $j = 4,5$, $rs_i + (r-1)\sum_{o=i+1}^{m} s_o + (r-1)\sum_{o=1}^{j-1} s_o \geq s_j$ for $i = 3$ and $j = 1,2$ (Condition (2) of Theorem 5), worker 1 has departed from the end of stations 4, 5, 1, and 2 when worker 2 reaches the start of stations 4, 5, 1, and 2. Thus, $b_4$, $b_5$, $b_1$, and $b_2$ will never occur.

Continuing to consider the dynamics of the two workers, we can conclude that passings will not occur before worker 1 completes two items and then reaches the start of station 3 because $\frac{1}{v_1} > \frac{s_3+s_4}{v_2}$, $\frac{1+s_4}{v_1} > \frac{s_3+s_4+s_5}{v_2}$, $\frac{1+s_4+s_5}{v_1} > \frac{s_3+s_4+s_5+s_1}{v_2}$, $\frac{1+s_4+s_5+s_1}{v_1} > \frac{1}{v_2}$, and $\frac{2-s_3}{v_1} > \frac{1+s_3}{v_2}$ (Condition (3) of Theorem 5). Because $\frac{2}{v_1} > \frac{1+s_3+s_4}{v_2}$ and $\frac{2+s_4}{v_1} > \frac{1+s_3+s_4+s_5}{v_2}$ ( $k + r - kr - rs_j > rs_i + (r-1)\sum_{o=i+1}^{j-1} s_o$ for $k = 2$, $i = 3$ and $j = 4,5$ (Condition (2) of Theorem 5), worker 2 has departed from the end of stations 4 and 5 when worker 1 completes two items and reaches the start of stations 4 and 5. Because $\frac{2+s_4+s_5}{v_1} > \frac{1+s_3+s_4+s_5+s_1}{v_2}$ and $\frac{2+s_4+s_5+s_1}{v_1} > \frac{2}{v_2}$ ( $k + r - kr - rs_j > rs_i + (r-1)\sum_{o=i+1}^{m} s_o + (r-1)\sum_{o=1}^{j-1} s_o$ for $k = 2$, $i = 3$ and $j = 1,2$ (Condition (2) of Theorem 5), worker 2 has departed from the end of stations 1 and 2 when worker 1 completes three items and reaches the start of stations 1 and 2. So, $p_4$, $p_5$, $p_1$, $p_2$ will not occur.

Because $\frac{3-s_3}{v_1} \leq \frac{2+s_3}{v_2}$ ($k + 1 - kr \leq (r+1)s_i$ for $k = 2$ and $i = 3$ (Condition (3) of Theorem 5), worker 1 completes three items and reaches the start of station 3 before worker 2 has completed two items and departed from the end of station 3. Thus, $p_3$ will occur. So, the cycle $p_3c_1c_2c_1c_2c_1p_3$ is constructed, where $k = 2$. Furthermore, when $s_3 = \frac{3-2r}{r+1} = 0.22 \Leftrightarrow \frac{3-s_3}{v_1} = \frac{2+s_3}{v_2}$, there is no waiting time for worker 1 when $p_3$ occurs and the maximal throughput is obtained, that is $v_1 + v_2 = 4.5$.

Example 9:

Figure A.8 is used to show that a period-3 cycle exists in a three-worker, four-station rotating *seru*. We set $v_1 = 2.0$, $v_2 = 1.5$, $v_3 = 1.0$, $x_1 = 0$, $x_2 = 0.35$, $x_3 = 0.05$, $s_1 =$

0.05, $s_2 = 0.3$, $s_3 = 0.6$, $s_4 = 0.05$. The time for worker 1 to arrive at the start of station 2 is $t_1 = \frac{0.05}{2.0} = 0.025$; the time for worker 3 to arrive at the end of station 2 is $t_2 = \frac{0.3}{1.0} = 0.3$. Since $t_1 < t_2$, we have $a^{(0)} = p_2^{1-3}$. After $p_2^{1-3}$, worker 2 is still working on station 3; then we have $a^{(1)} = p_3^{1-2}$. The time for worker 1 to complete an item and then arrive at the start of station 2 is $t_1^{'} = \frac{0.6}{1.5} + \frac{0.05}{2.0} + \frac{0.05}{2.0} = 0.45$; the time for worker 3 to arrive at the end of station 2 is $t_3^{'} = 0.3 + \frac{0.3}{1.0} = 0.6$. Since $t_1^{'} < t_3^{'}$, we have $a^{(2)} = p_2^{1-3}$. After $p_2^{1-3}$, worker 2 is still working on station 3 and the time for worker 2 to arrive at the end of station 3 is $t_2^{'} = \frac{0.6}{1.5} + \frac{0.6}{1.5} = 0.8$; then we have $a^{(3)} = p_3^{1-2}$. The time for worker 1 to complete an item and then arrive at the start of station 2 is $t_1^{''} = 0.8 + \frac{0.05}{2.0} + \frac{0.05}{2.0} = 0.85$; the time for worker 3 to arrive the end of station 2 is $t_3^{''} = 0.6 + \frac{0.3}{1.0} = 0.9$. Since $t_1^{''} < t_3^{''}$, we have $a^{(4)} = p_2^{1-3}$. After $p_2^{1-3}$, worker 2 is still working on station 3; then we have $a^{(5)} = p_3^{1-2}$. The time for worker 3 to arrive at the start of station 3 is $0.9 + \frac{0.3}{0.1} = 1.2$; the time for worker 2 to arrive at the end of station 3 is $0.8 + \frac{0.6}{1.5} = 1.2$. That is, worker 3 arrives at the start of station 3 when worker 2 arrives at the end of station 3. After $p_3^{1-2}$, worker 3 is blocked at the start of station 3; then we have $a^{(6)} = b_3^{3-2}$. The time for worker 1 to complete an item and arrive at the start of station 2 is $1.2 + \frac{0.05}{2.0} + \frac{0.05}{2.0} = 1.25$; at this time, worker 3 is still blocked at the start of station 3 by worker 2. Thus, we have $a^{(7)} = p_2^{1-3}$. After $p_2^{1-3}$, worker 2 is still working on station 3; then we have $a^{(8)} = p_3^{1-2}$. Continuing to study the dynamics of these three workers, we have $a^{(9)} = b_3^{3-2}$, and the orbit $p_2^{1-3} \to p_3^{1-2} \to p_2^{1-3} \to p_3^{1-2} \to p_2^{1-3} \to p_3^{1-2} \to b_3^{3-2} \to p_2^{1-3} \to p_3^{1-2} \to b_3^{3-2} \ldots$ is obtained.

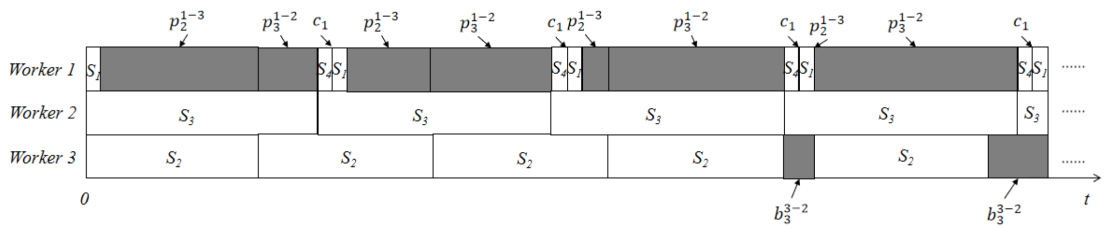



Figure A.8. An example of three workers and four stations

In this period-3 cycle $b_3^{3-2} \to p_2^{1-3} \to p_3^{1-2} \to b_3^{3-2}$, passing and blocking behaviors occur at stations 2 and 3, which form a continuous interval $s_2 + s_3$. So, as stated in the Li-Yorke Theorem, this higher dimensional rotating *seru* has chaotic characteristics.